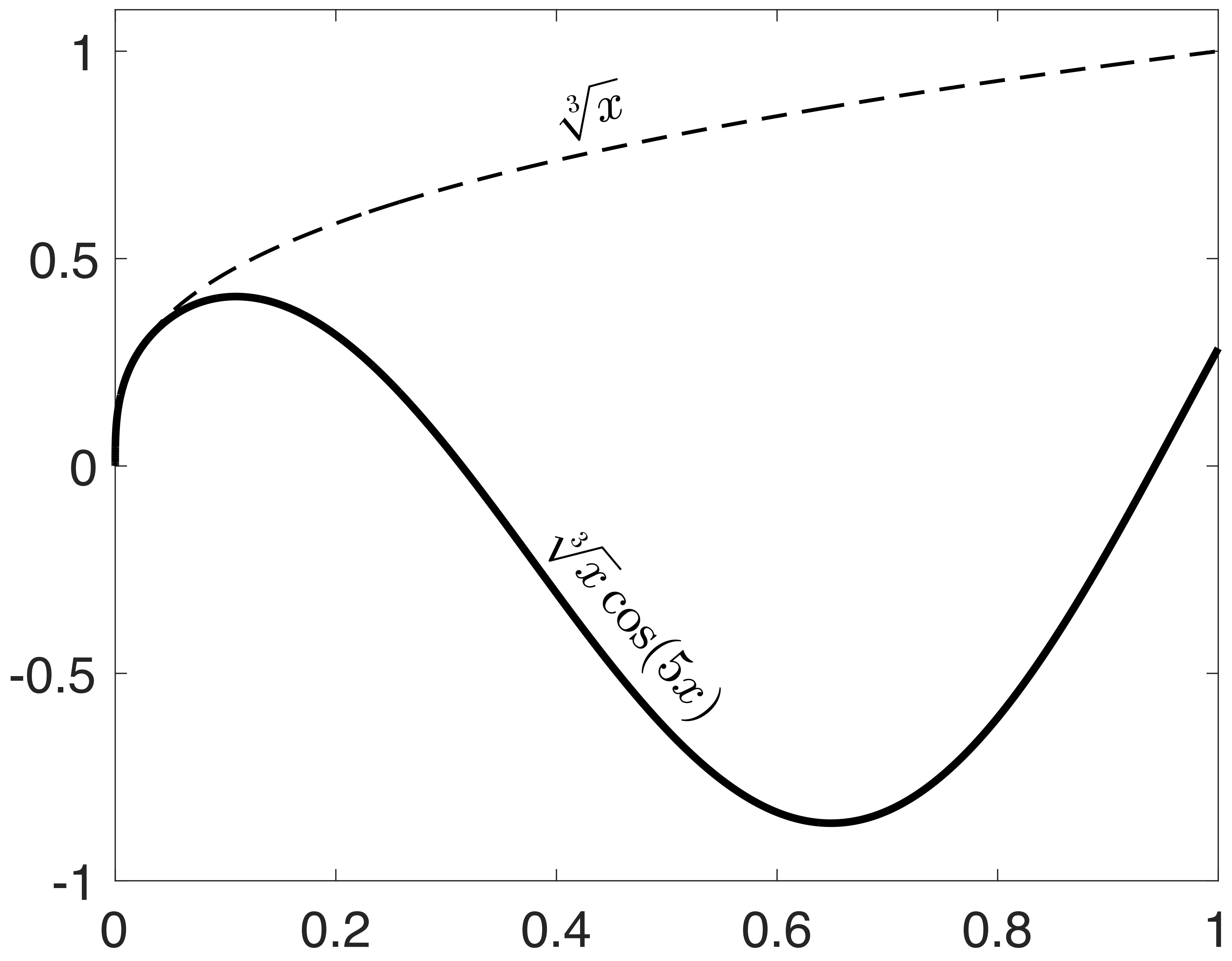
KU LEUVEN

Rational approximation using partial fractions with preassigned poles

Astrid Herremans
joint work with Daan Huybrechs, Nick Trefethen and Nicolas Boullé

NUMA fwo

- the univariate case
- the multivariate case
- on the partial fractions representation



1
0.5
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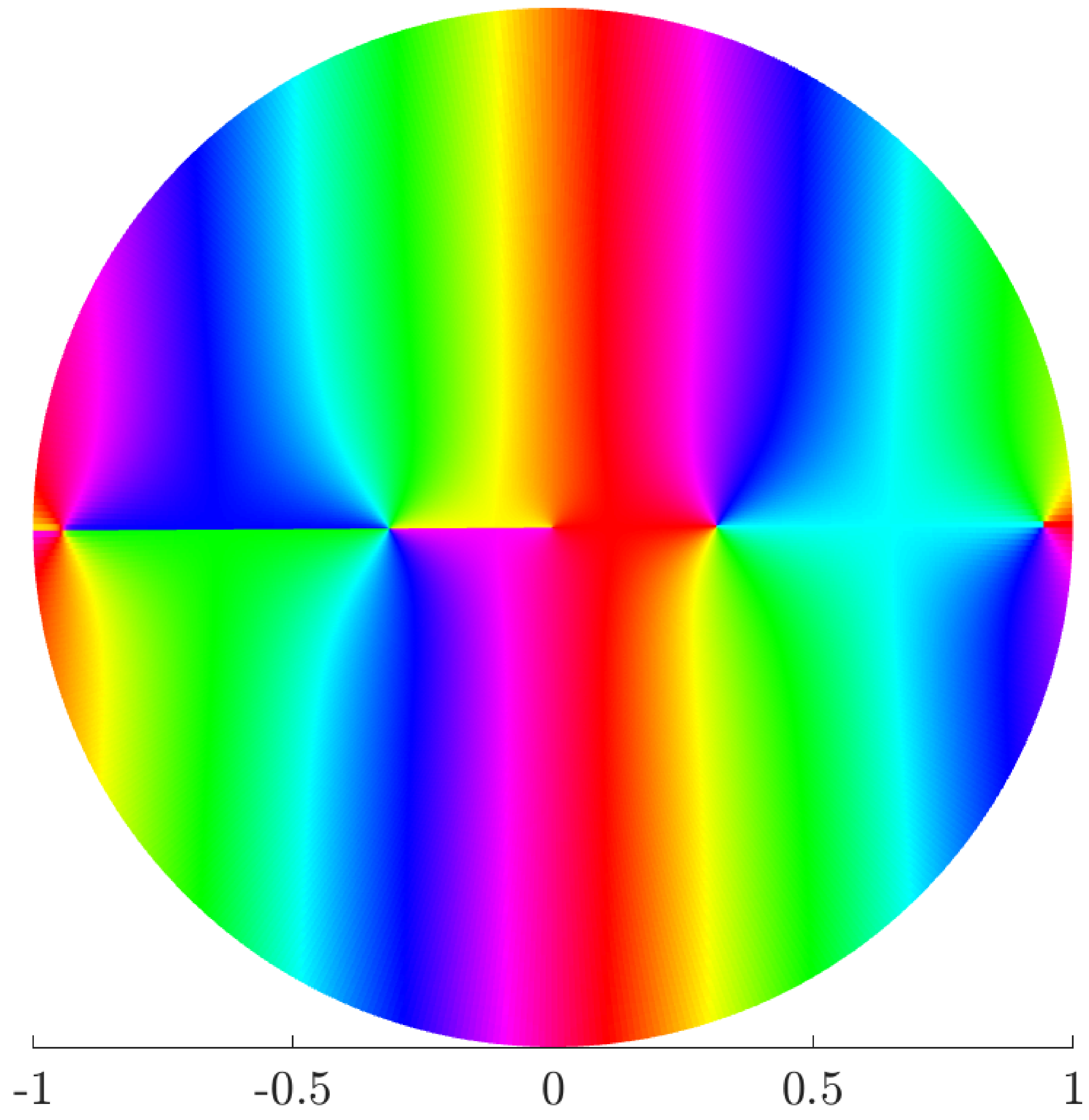
0.6
0.8
1
$\sqrt[3]{x}$
$\sqrt[3]{x}\cos(5x)$



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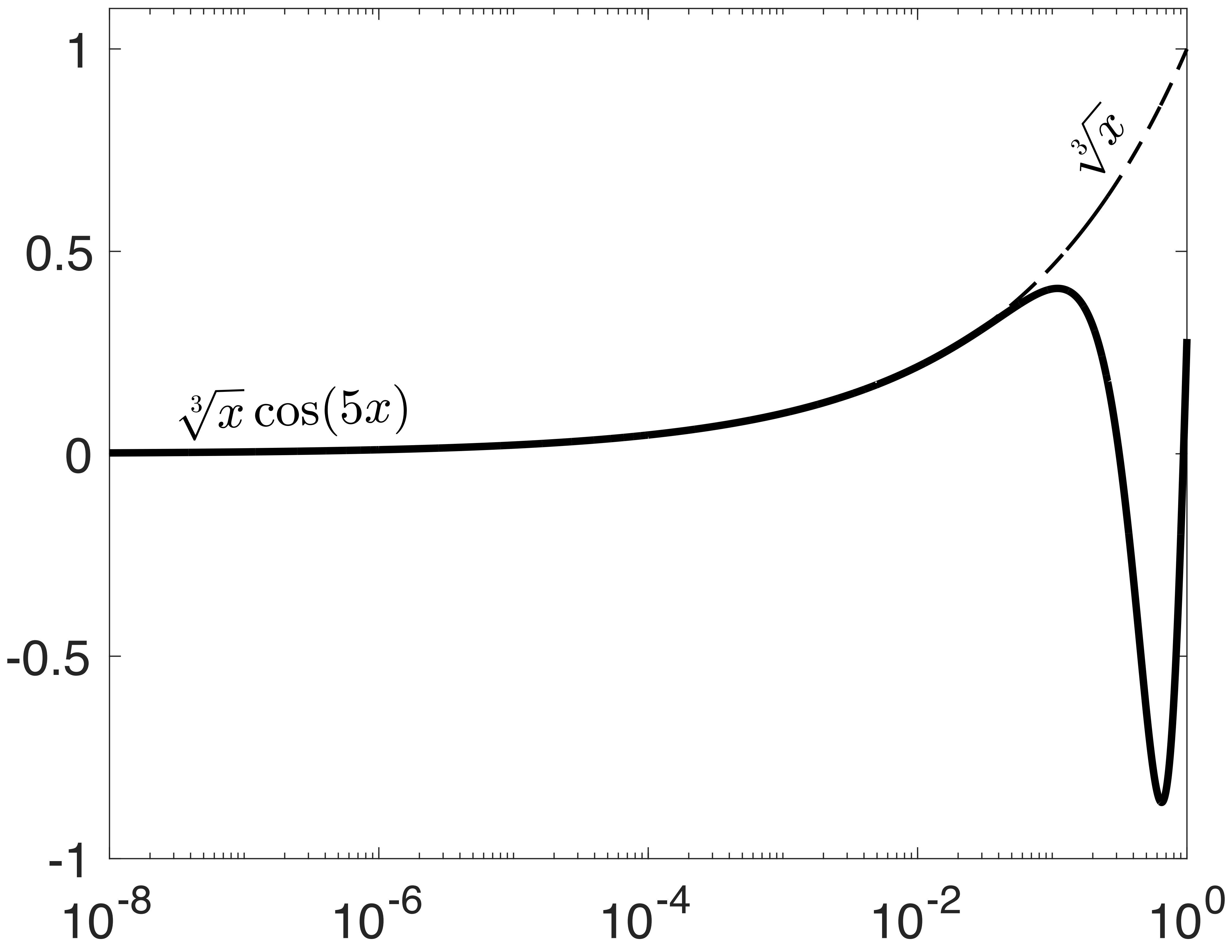

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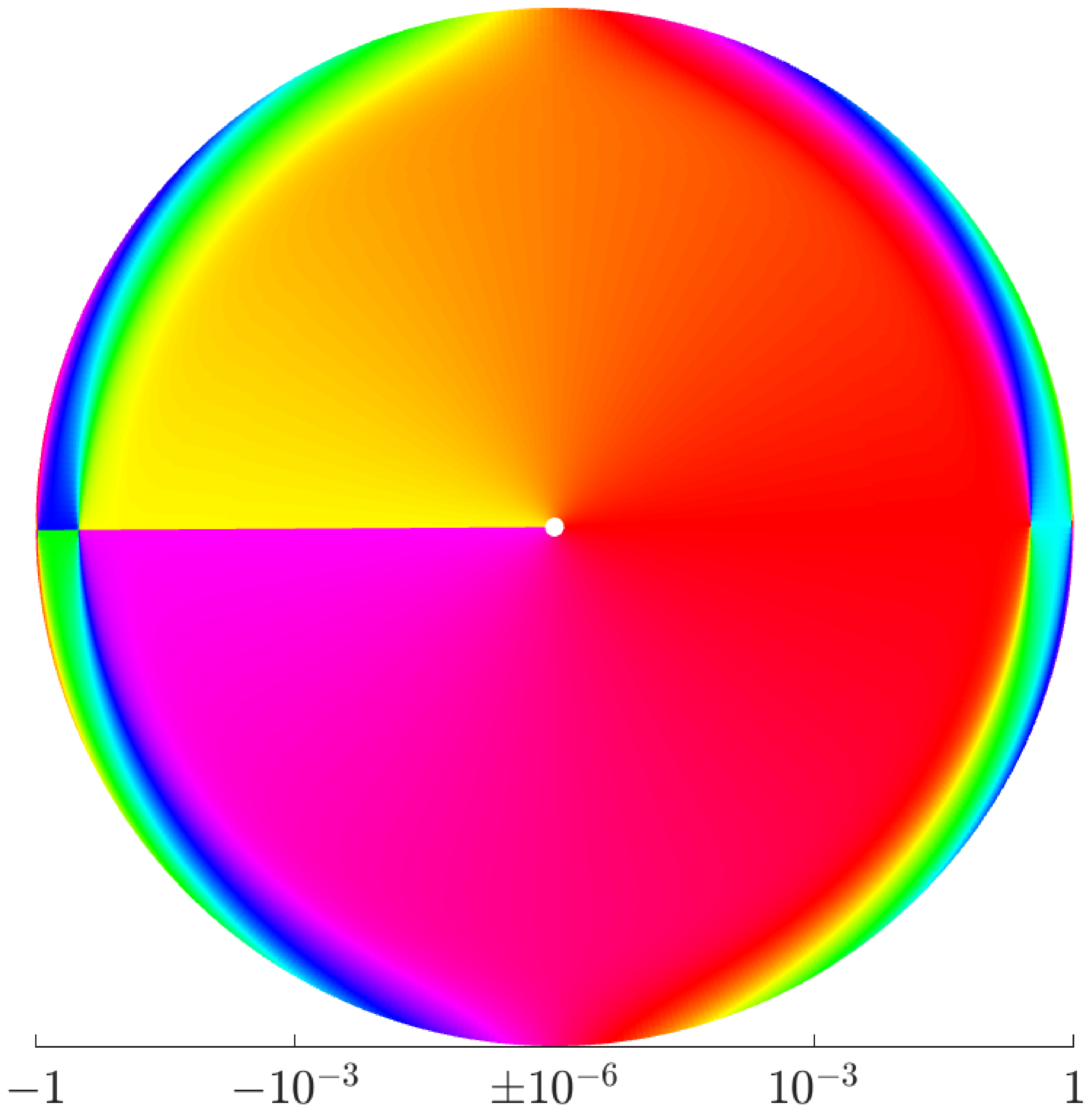

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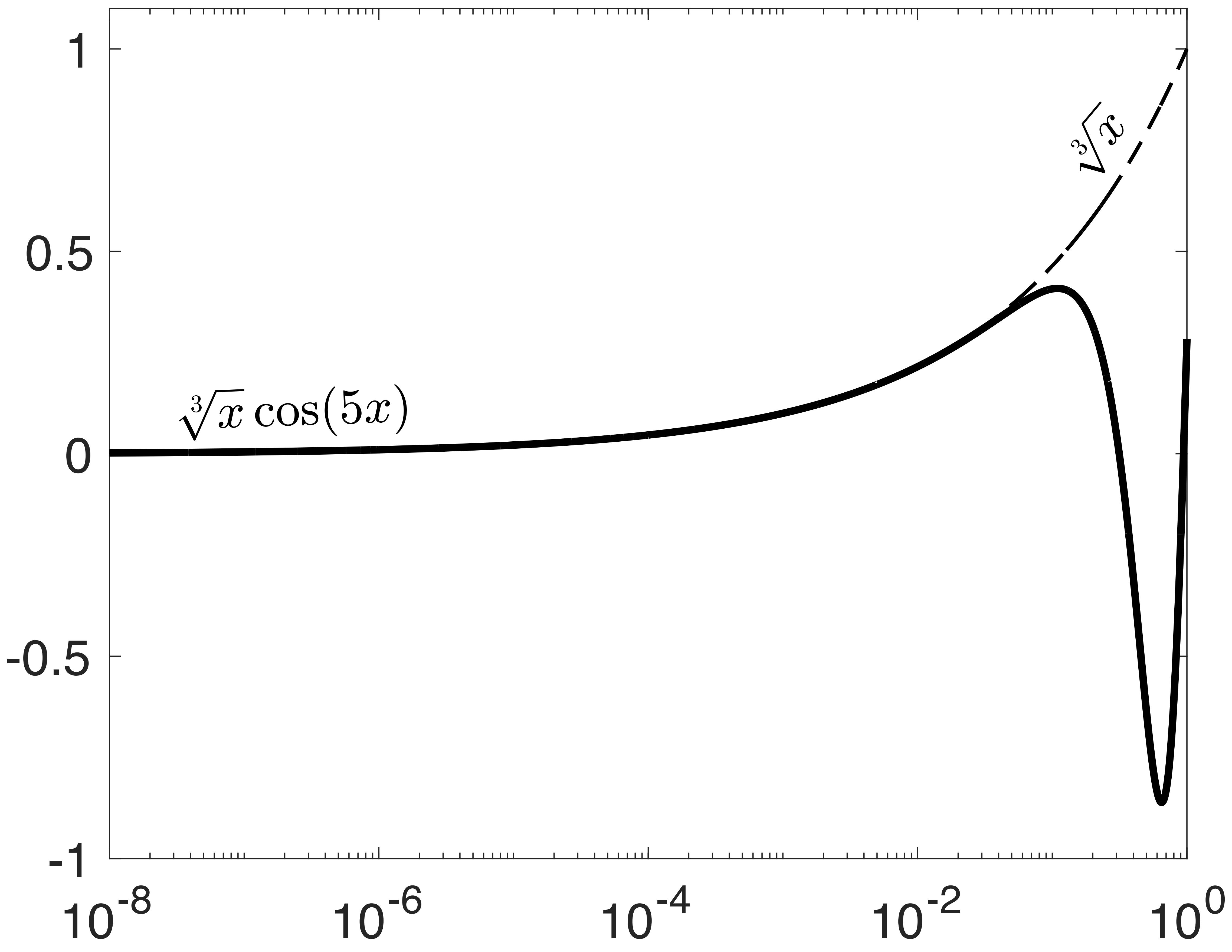

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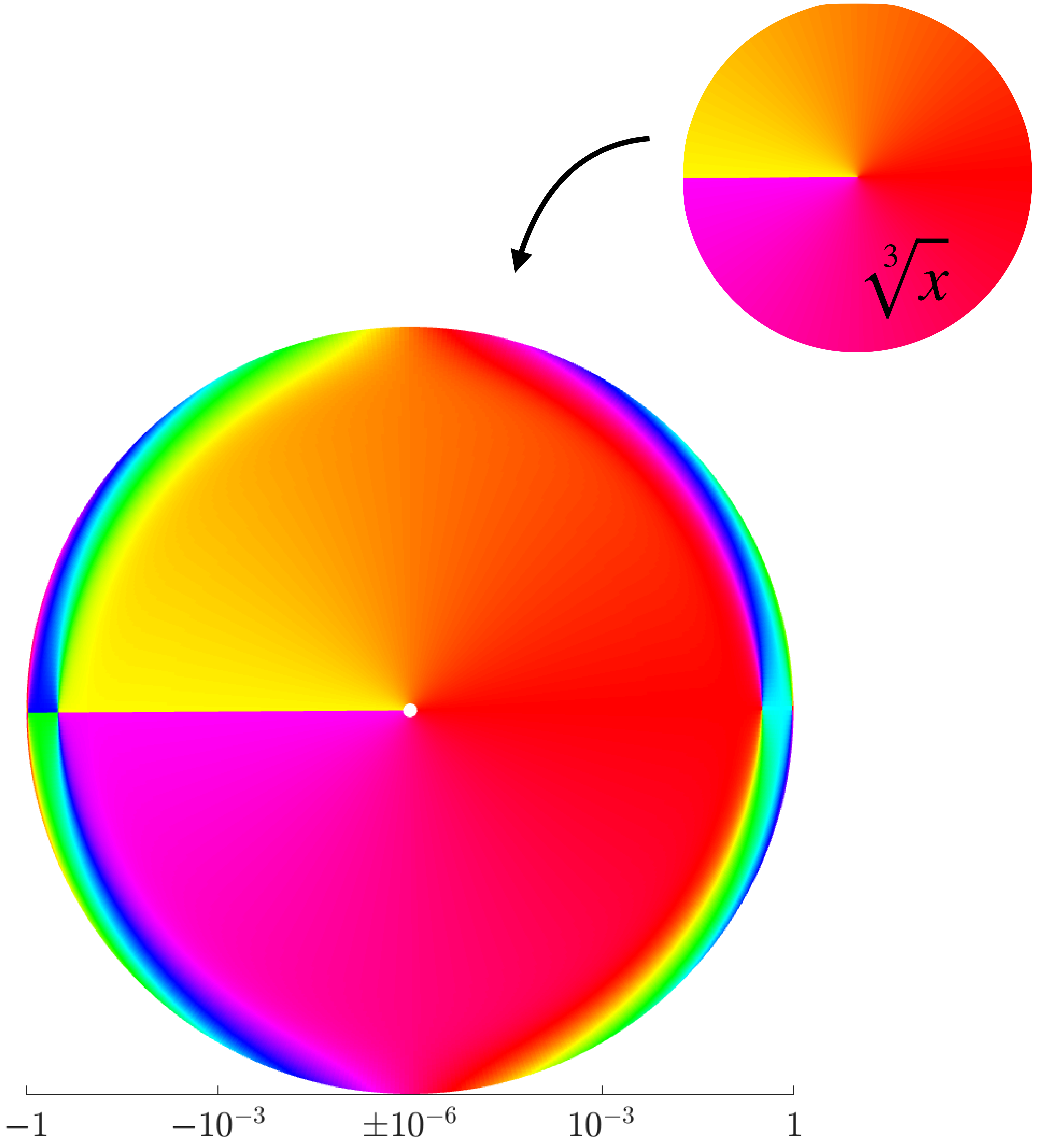

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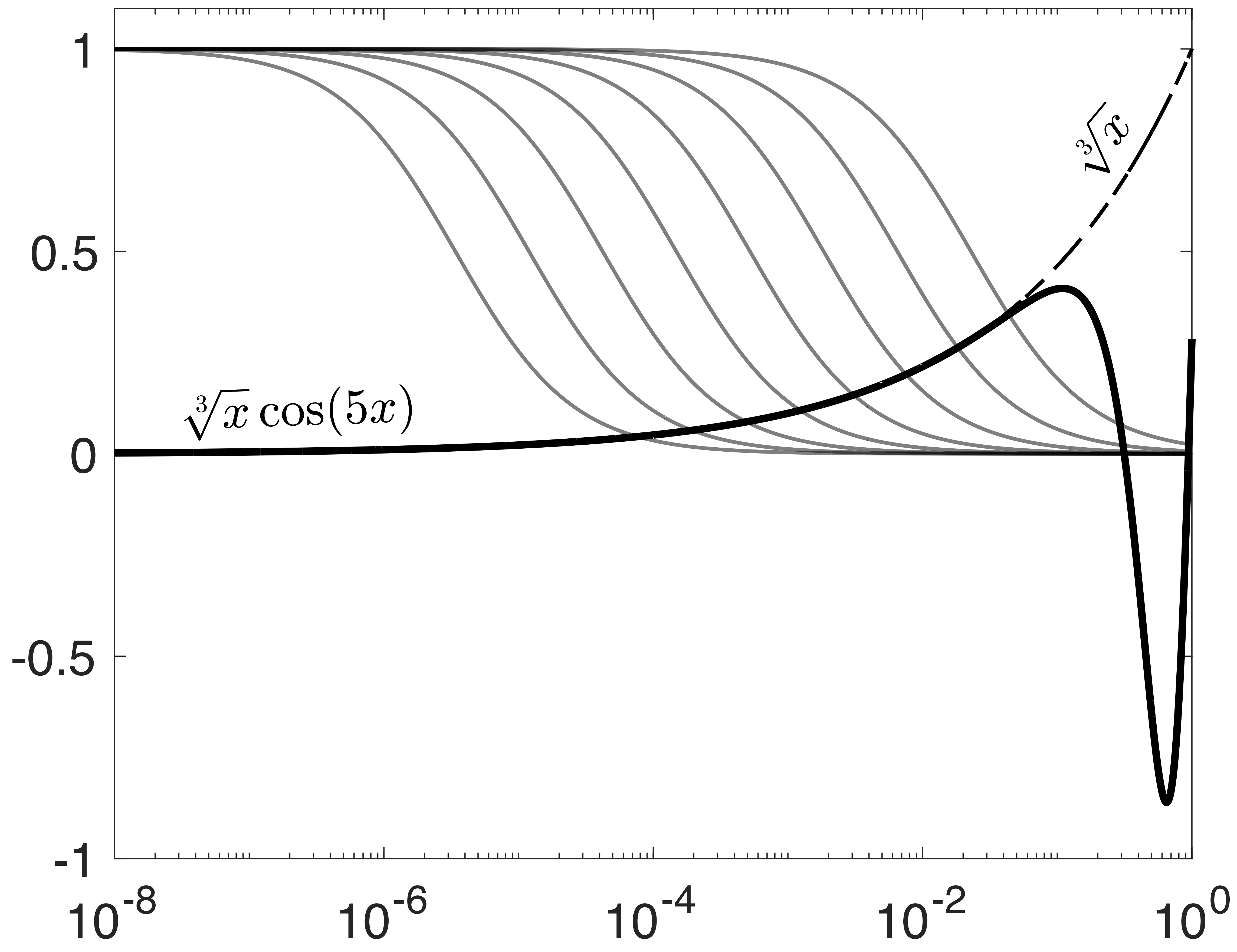

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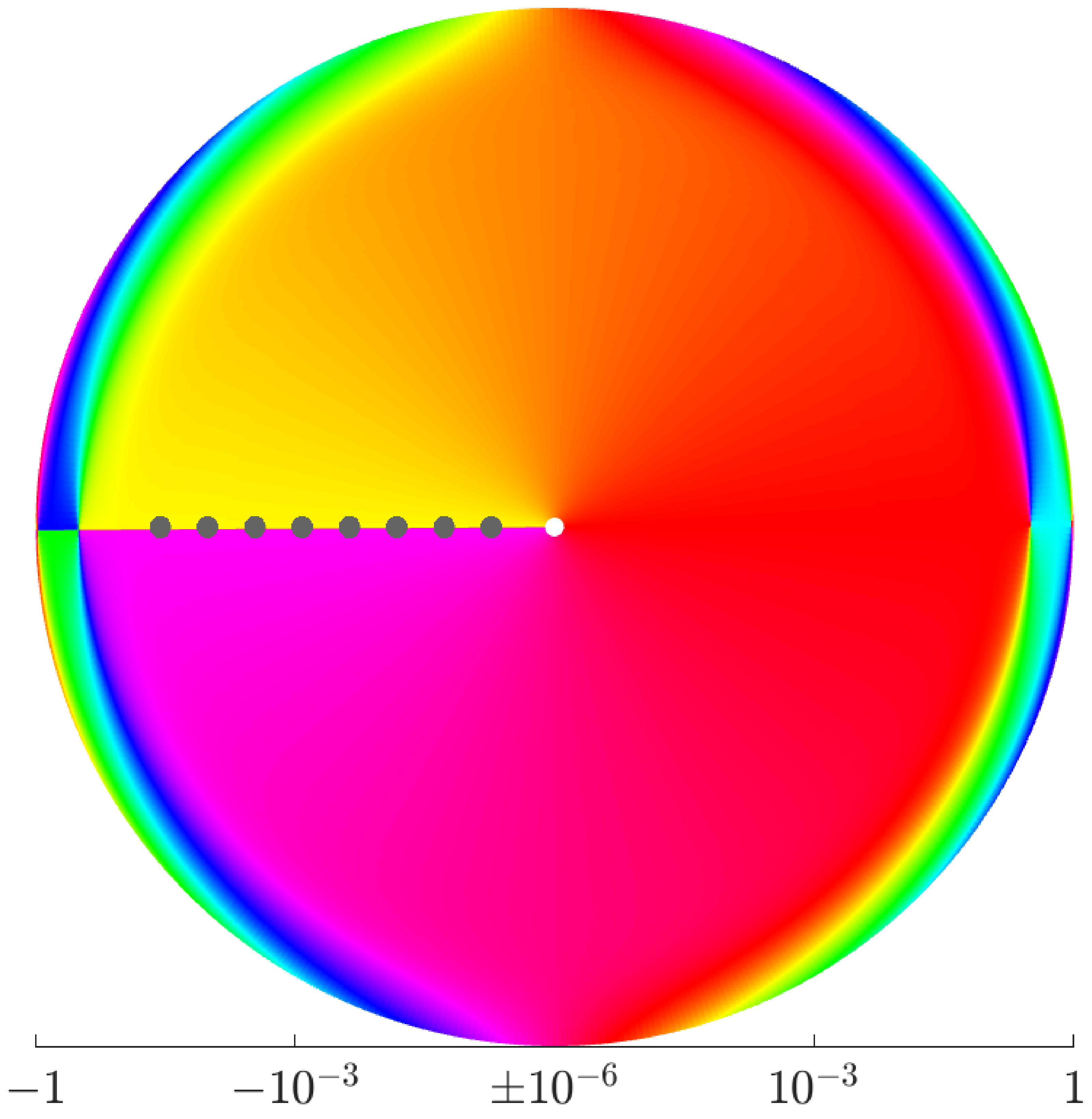

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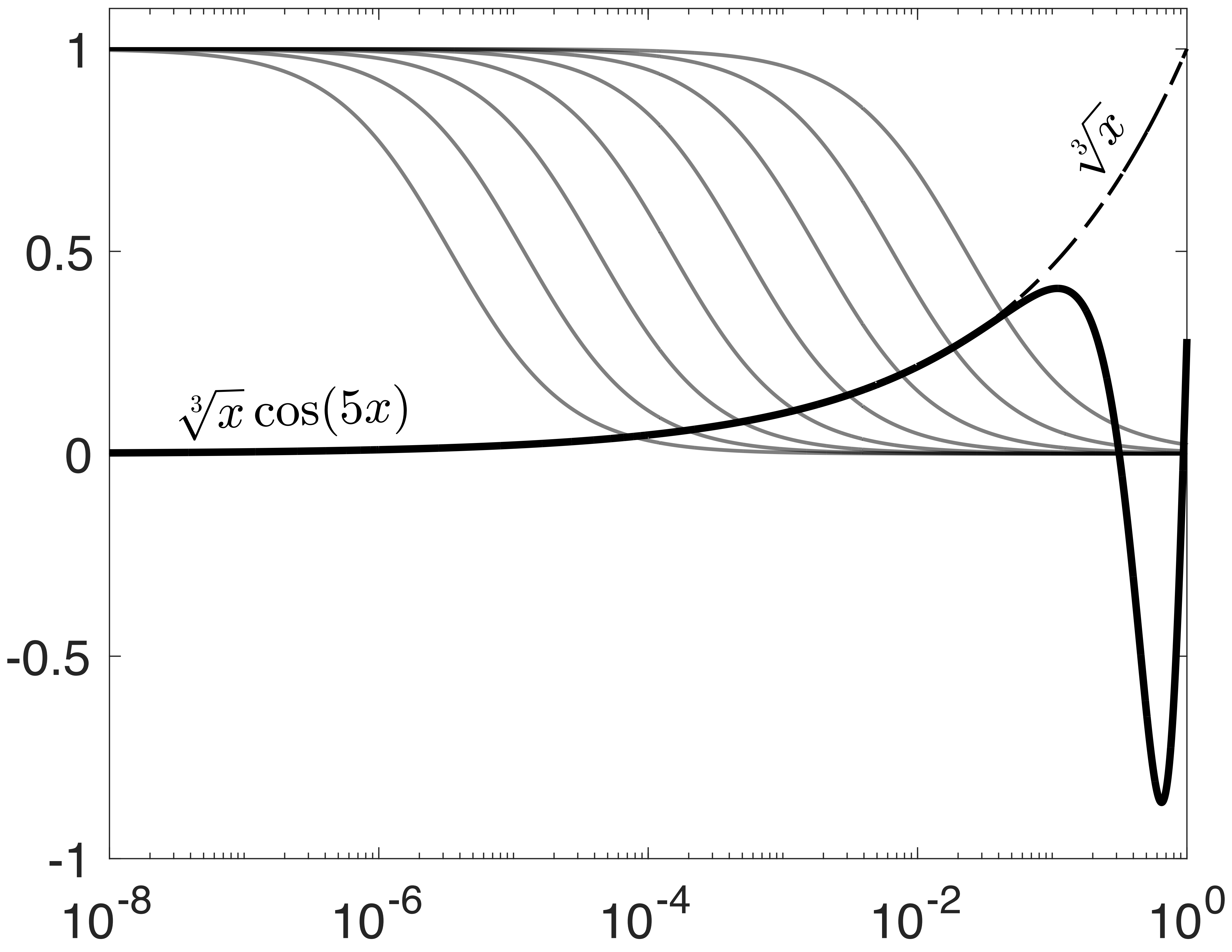
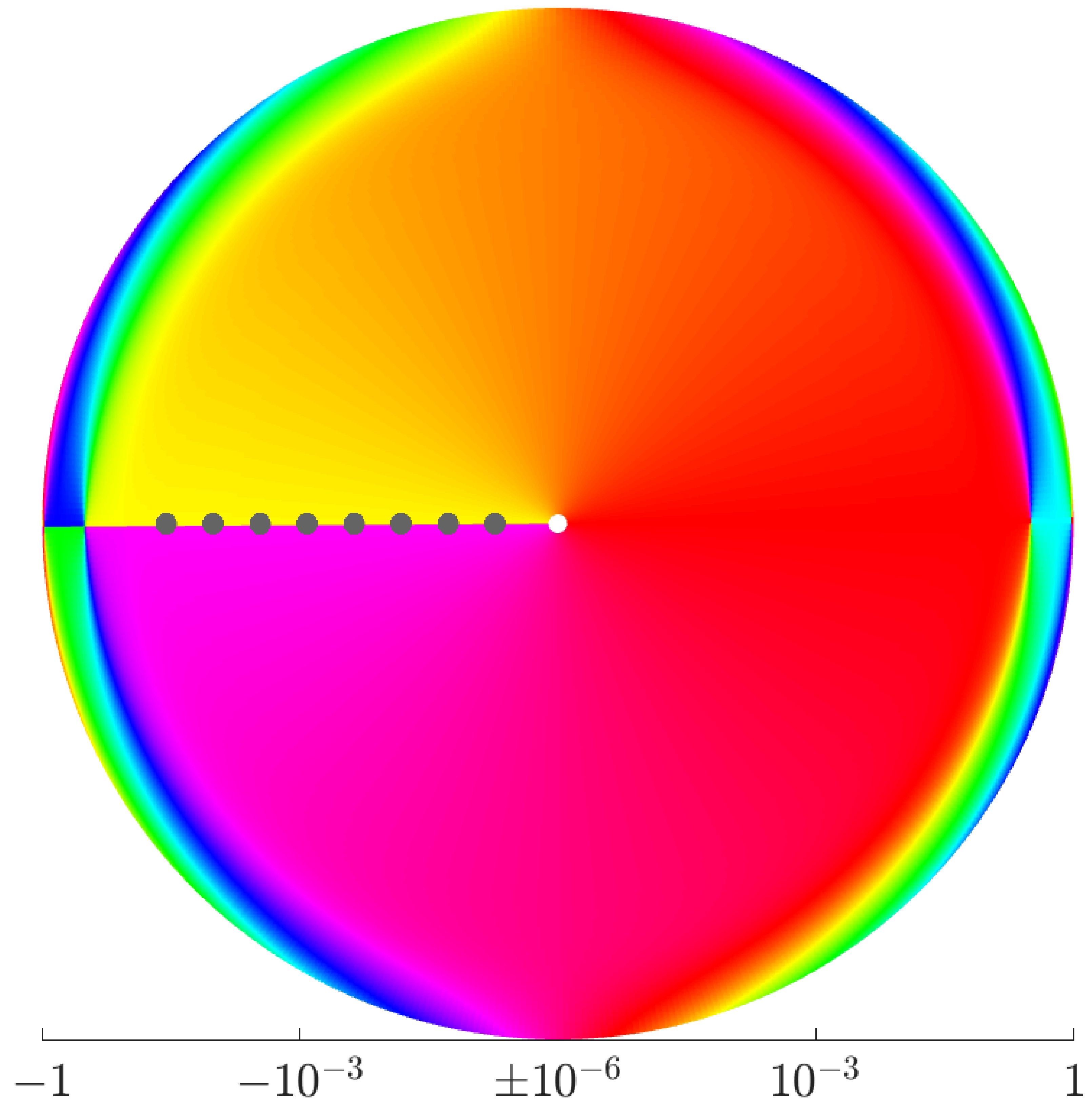
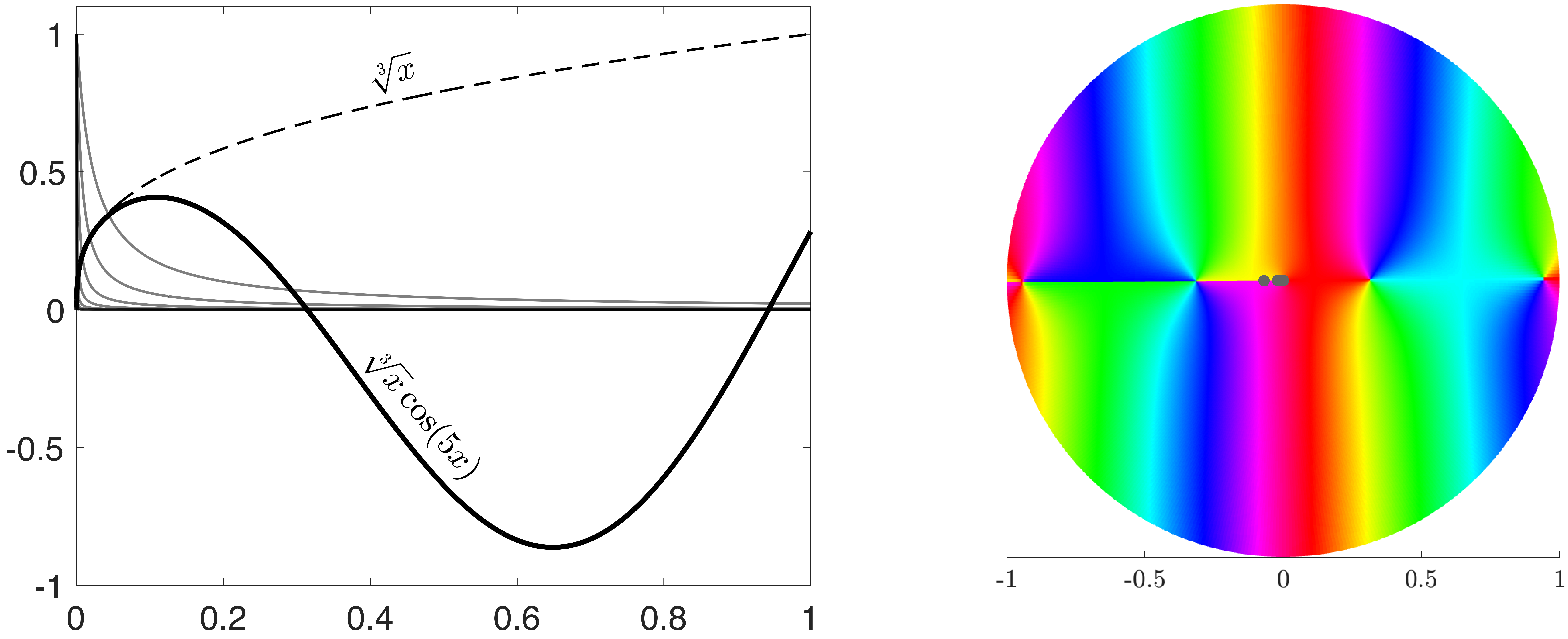
rational: $\dfrac{p_k}{x - p_k}$ $\quad \xleftrightarrow{s = \log x} \quad$ sigmoid: $\dfrac{1}{1 + e^{s - s_k}}$ (Huybrechs, Trefethen 2024)







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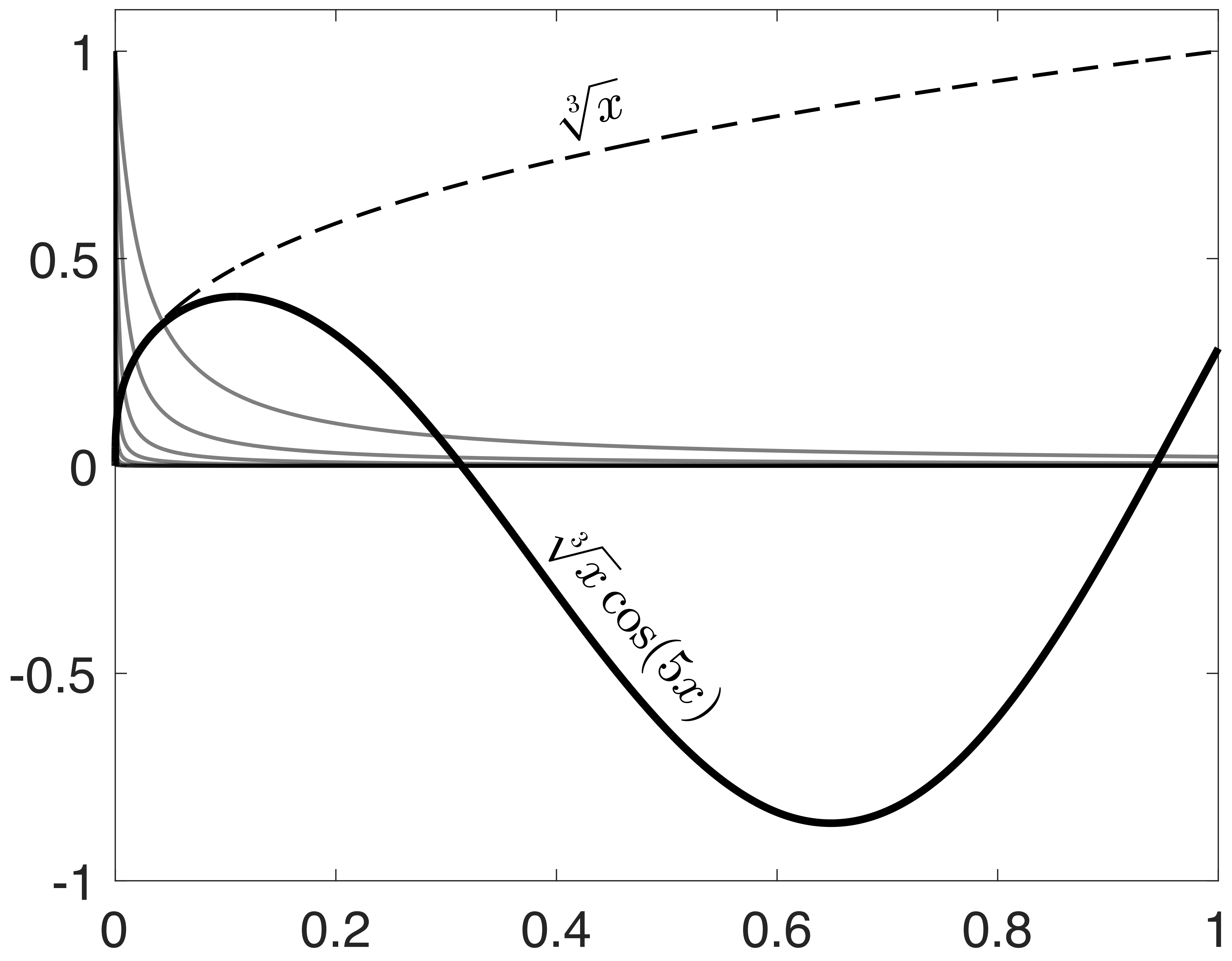
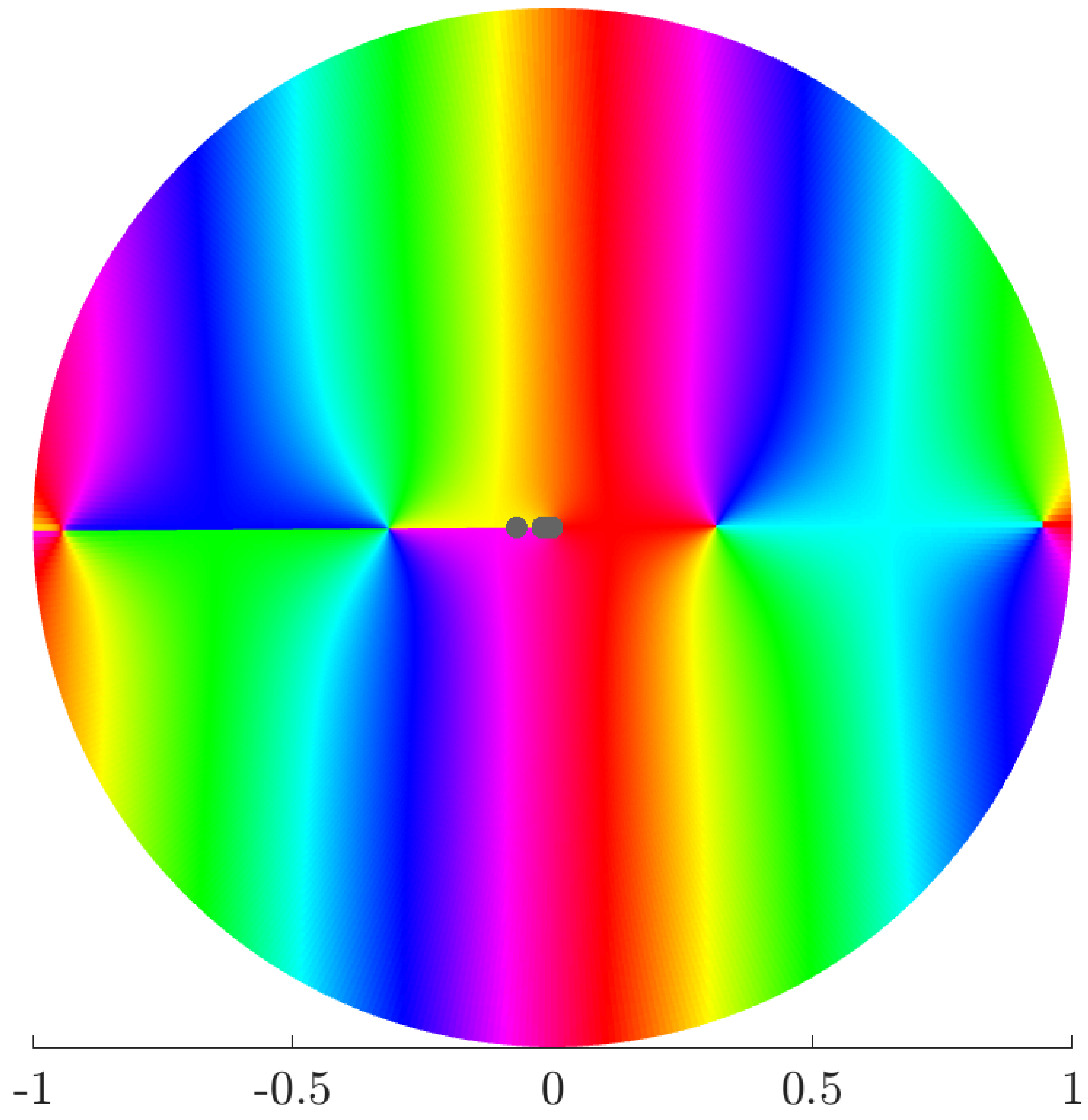
simply use big poles / polynomials to capture the smooth behaviour





Lightning approximation

Given the locations of the singularities $\{z_j\}$ of f,

$$f(z) \approx r(z) = \overbrace{\sum_{k=1}^{n_1} \frac{a_k p_k}{z - p_k}}^{\text{"LIGHTNING"}} + \overbrace{\sum_{k=0}^{n_2} b_k T_k(z)}^{\text{"POLYNOMIAL"}}$$

$f(z)$: $\mathcal{O}(|z - z_j|^{\delta_j})$ as $z \to z_j$

Lightning part: captures singular behaviour near z_j's

Polynomial part: captures smooth behaviour

Lightning approximation 

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$$f(z) \approx r(z) = \overbrace{\sum_{k=1}^{n_1} \frac{a_k p_k}{z - p_k}}^{\text{“LIGHTNING”}} + \overbrace{\sum_{k=0}^{n_2} b_k T_k(z)}^{\text{“POLYNOMIAL”}}$$

→ finding a_k and b_k can be done via least squares fitting

Root-exponential convergence

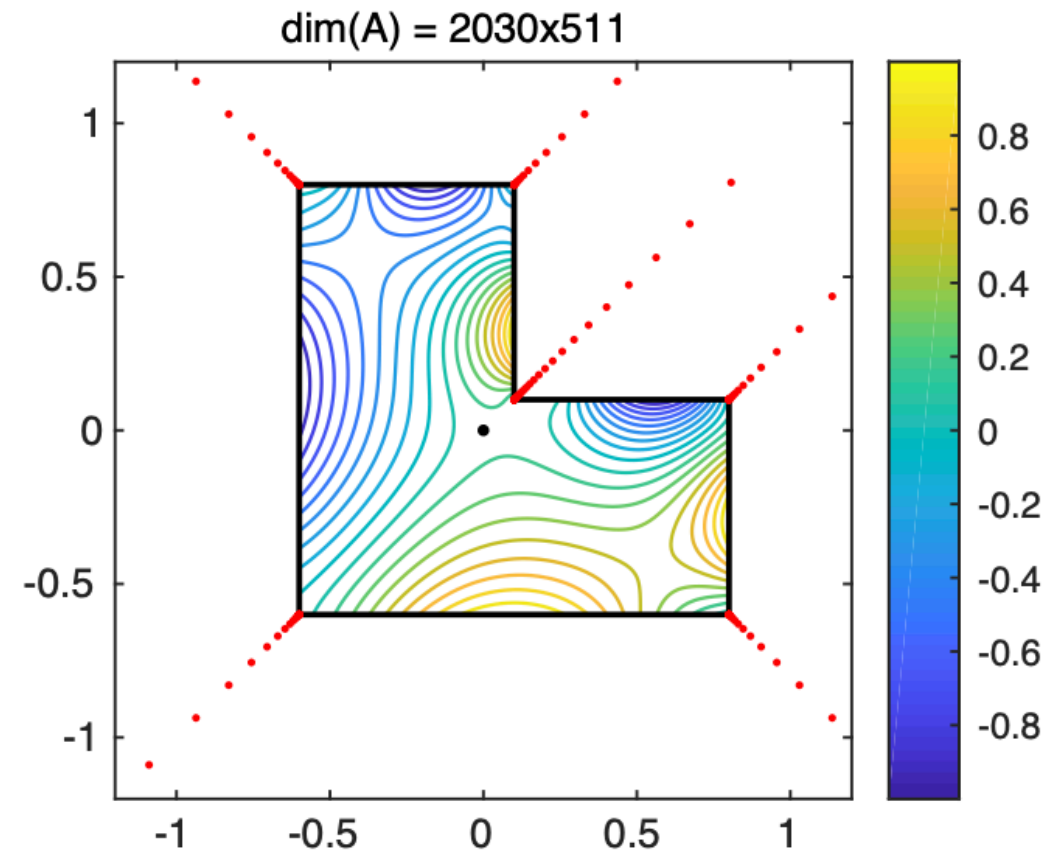
[Gopal and Trefethen, 2019](#)

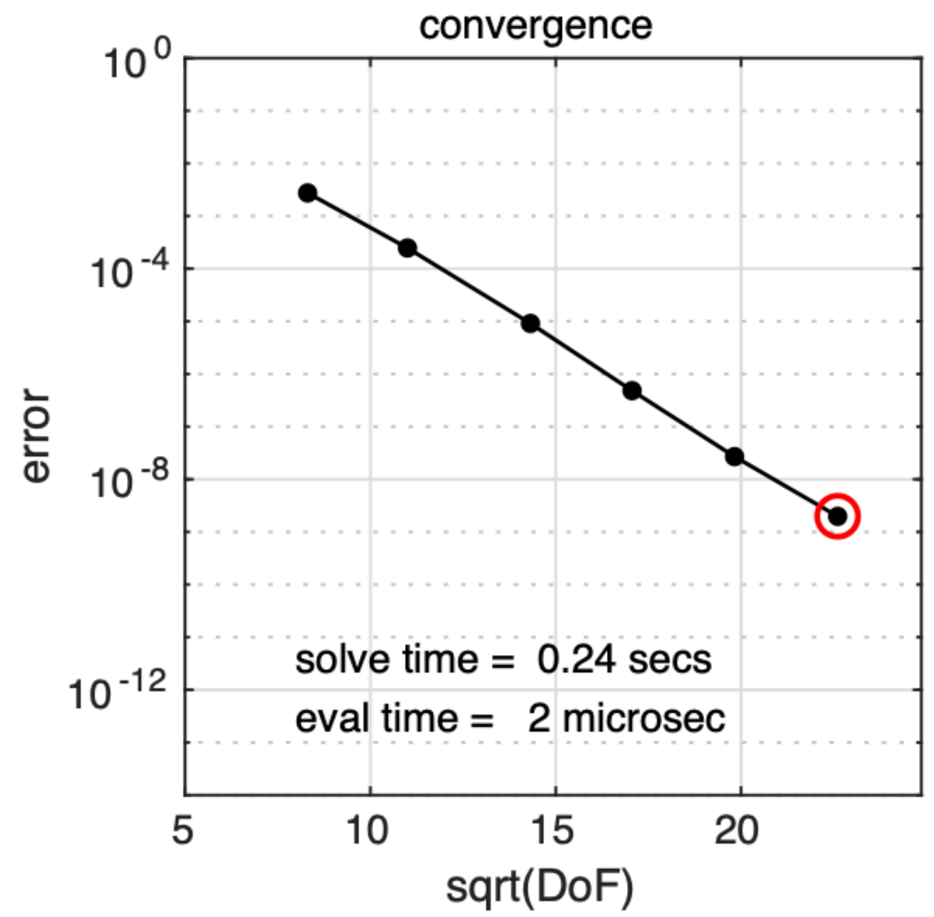
THEOREM 2.3. *Let Ω be a convex polygon with corners $w_1, \ldots, w_m$, and let f be an analytic function in Ω that is analytic on the interior of each side segment and can be analytically continued to a disk near each w_k with a slit along the exterior bisector there. Assume f satisfies $f(z) - f(w_k) = O(|z - w_k|^\delta)$ as $z \to w_k$ for each k for some $\delta > 0$. There exist degree n rational functions $\{r_n\}$, $1 \le n < \infty$, such that*

$$\|f - r_n\|_\Omega = O(e^{-C\sqrt{n}}) \tag{2.11}$$

as $n \to \infty$ for some $C > 0$. Moreover, each r_n can be taken to have finite poles only at points exponentially clustered along the exterior bisectors at the corners, with arbitrary clustering parameter σ as in (2.5), as long as the number of poles near each w_k grows at least in proportion to n as $n \to \infty$.

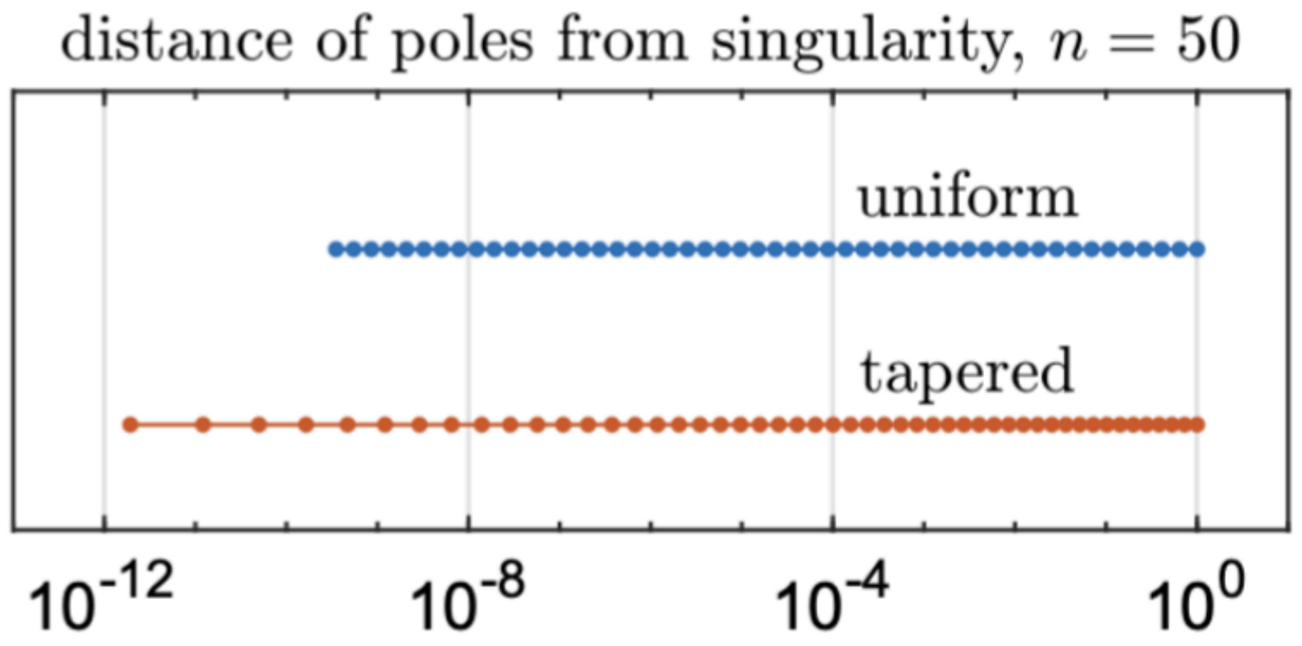
$$\beta_j = -e^{-\sigma j/\sqrt{n}}, \quad 0 \le j \le n - 1 \tag{2.5}$$



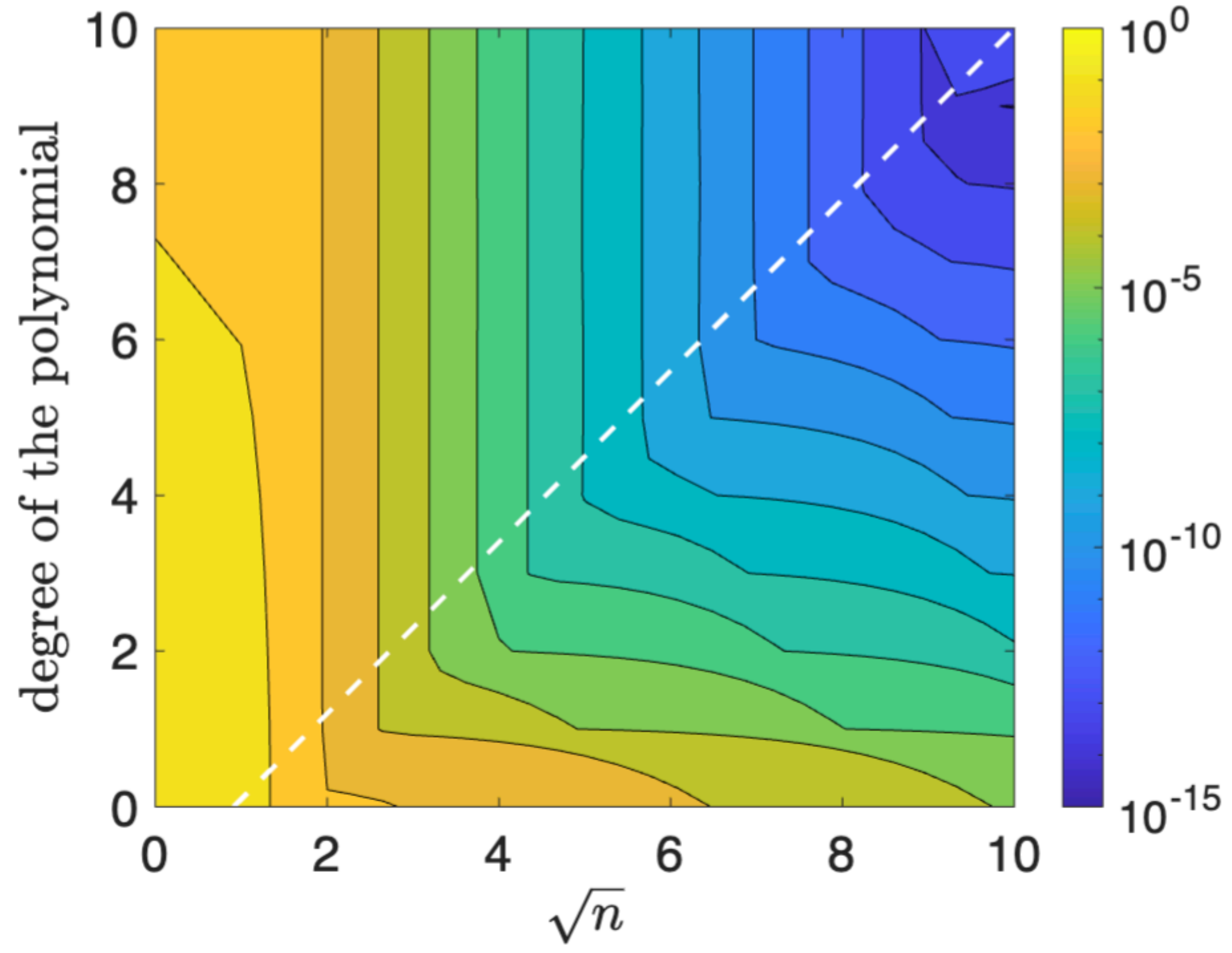


Optimisations

- "tapering" of the poles (Trefethen, Nakatsukasa and Weideman, 2021)



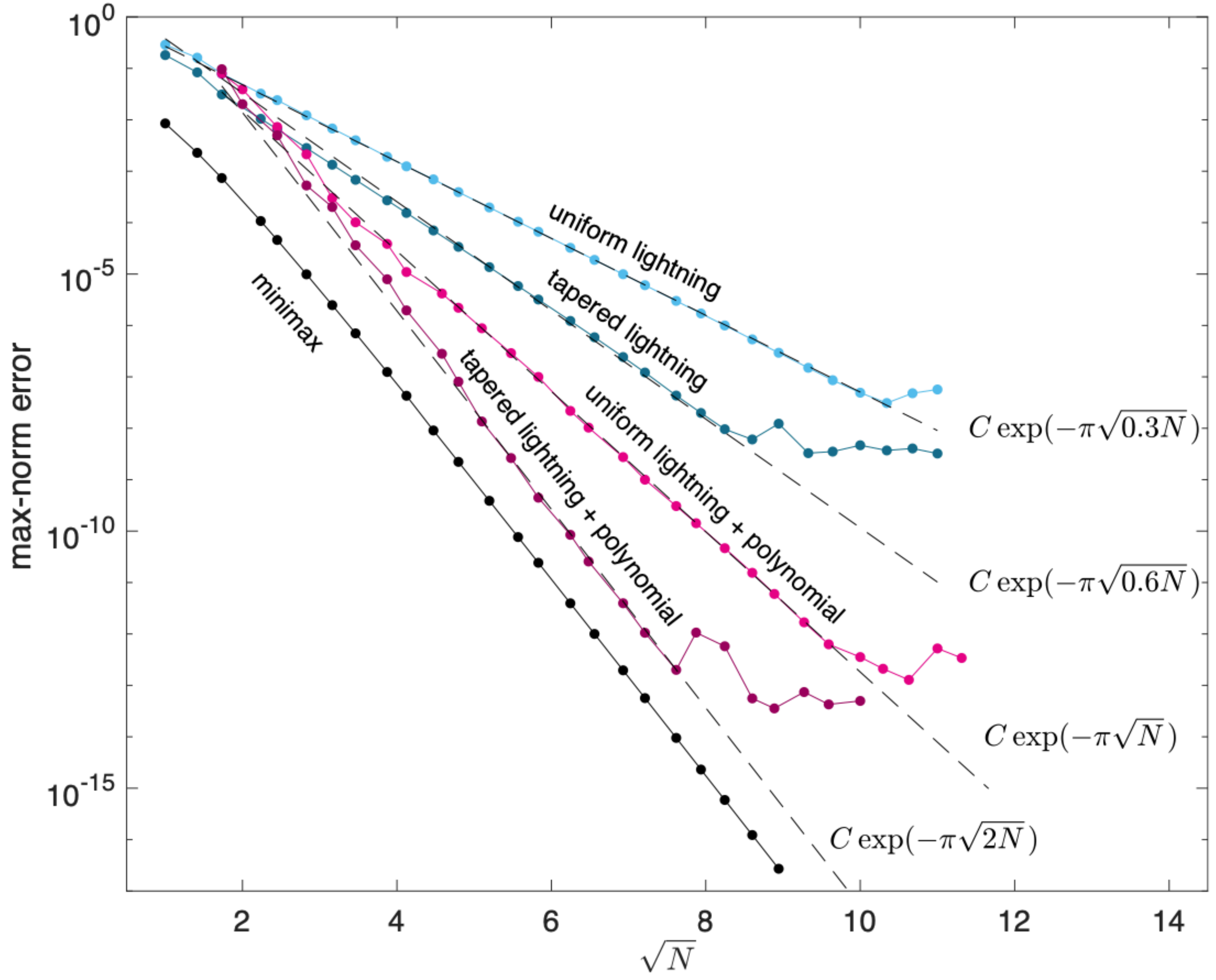
- adding a low-degree polynomial (H., Huybrechs and Trefethen, 2023)



- for singularities of type x^α, use $\sigma = 2\pi/\sqrt{\alpha}$

(H., Huybrechs and Trefethen, 2023) + (Xiang, Yang and Wu, 2024)

Optimal convergence rate



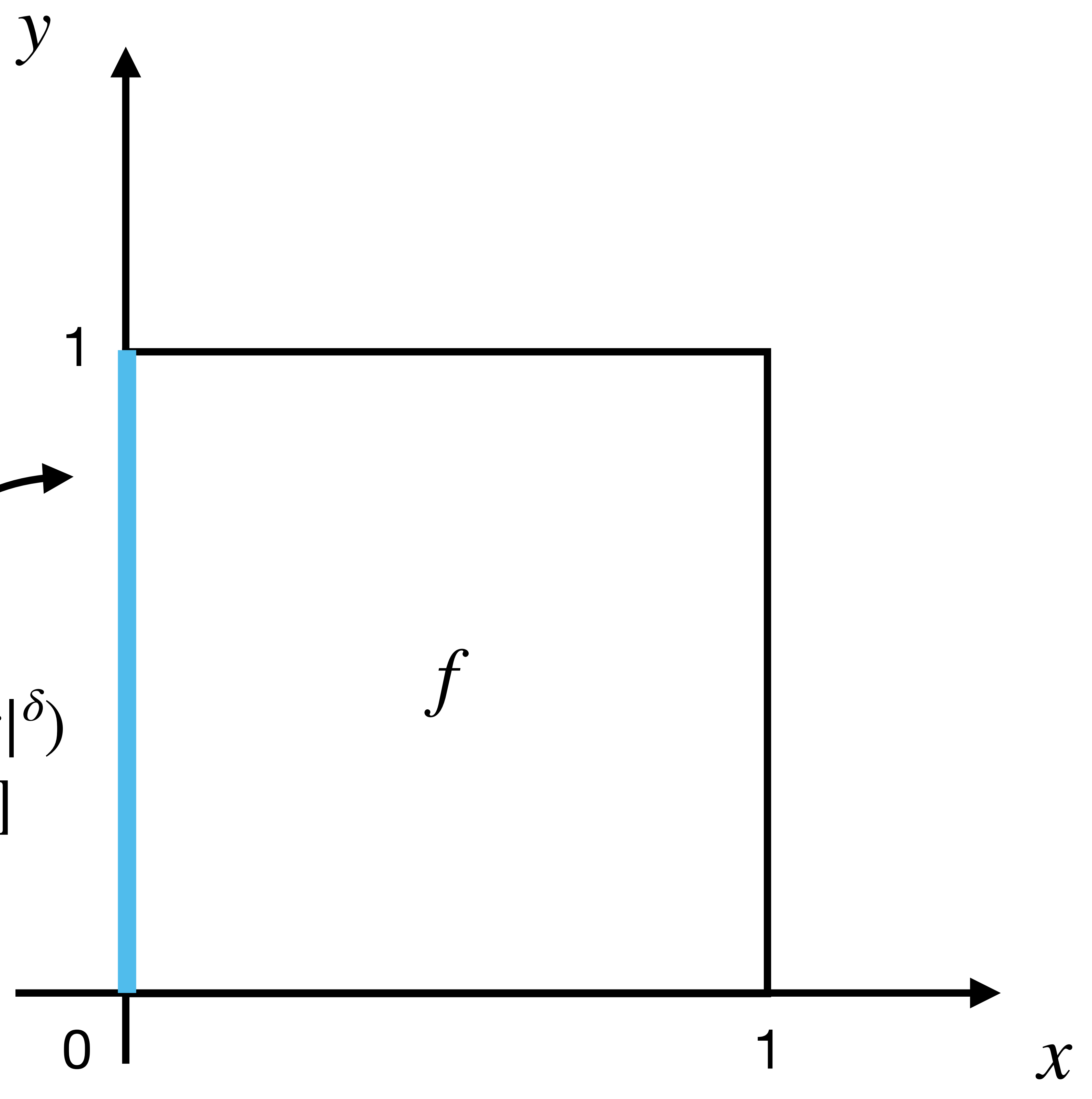
(H., Huybrechs and Trefethen, 2023)

Multivariate case

→ exploring multivariate lightning approximations (Boullé, H. and Huybrechs, 2024)

$$f(x,y) \approx \sum_k \frac{a_k(y)p_k}{x - p_k} + b(x,y)$$

$\exists \delta > 0 : \; f(x,y) = \mathcal{O}(|x|^{\delta})$
as $x \to 0, \;\; \forall y \in [0,1]$



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Remember: poles could be chosen independently of the type of singularity

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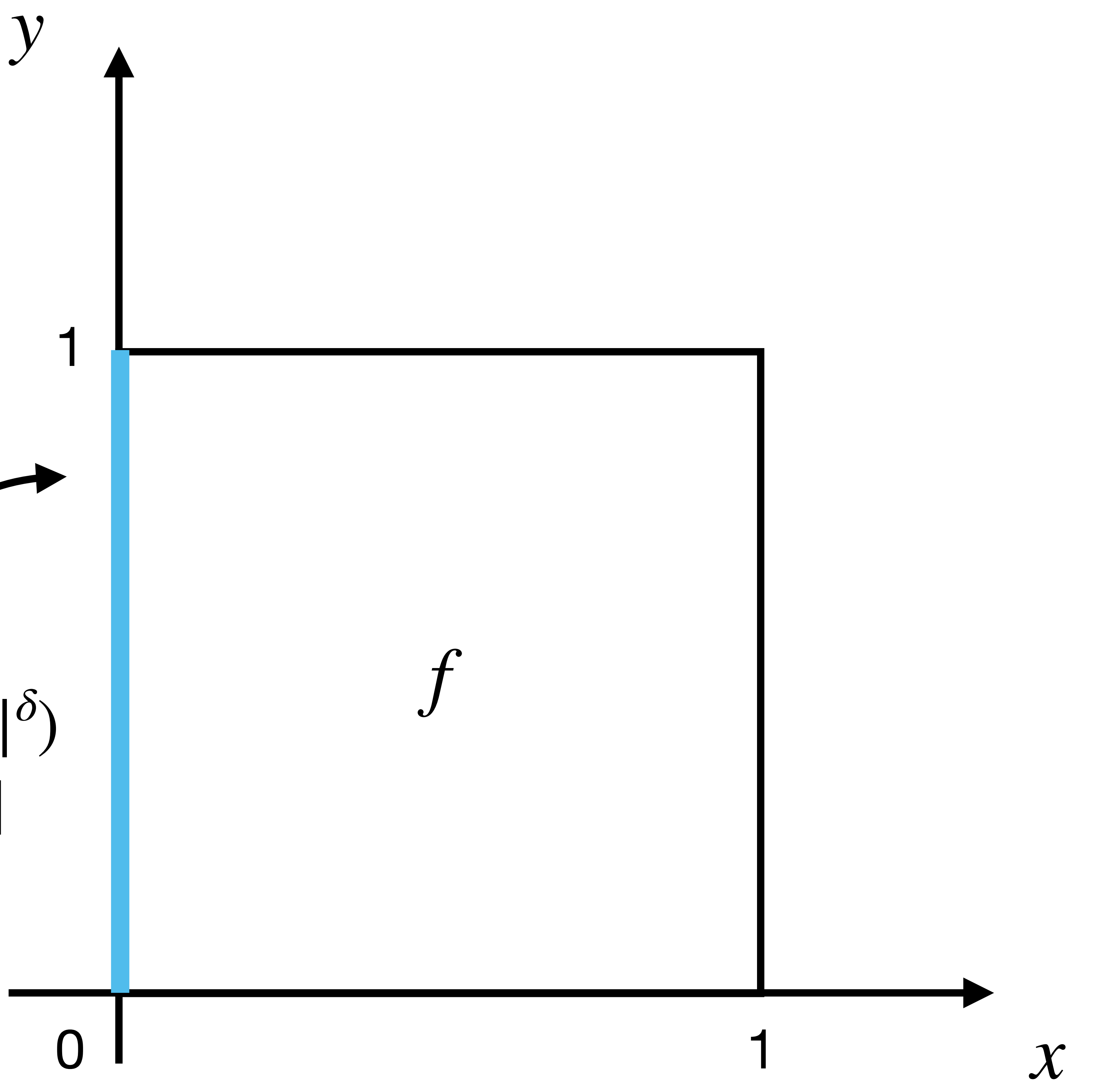

Multivariate case

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Variation along curve of singularities
→ low degree polynomial

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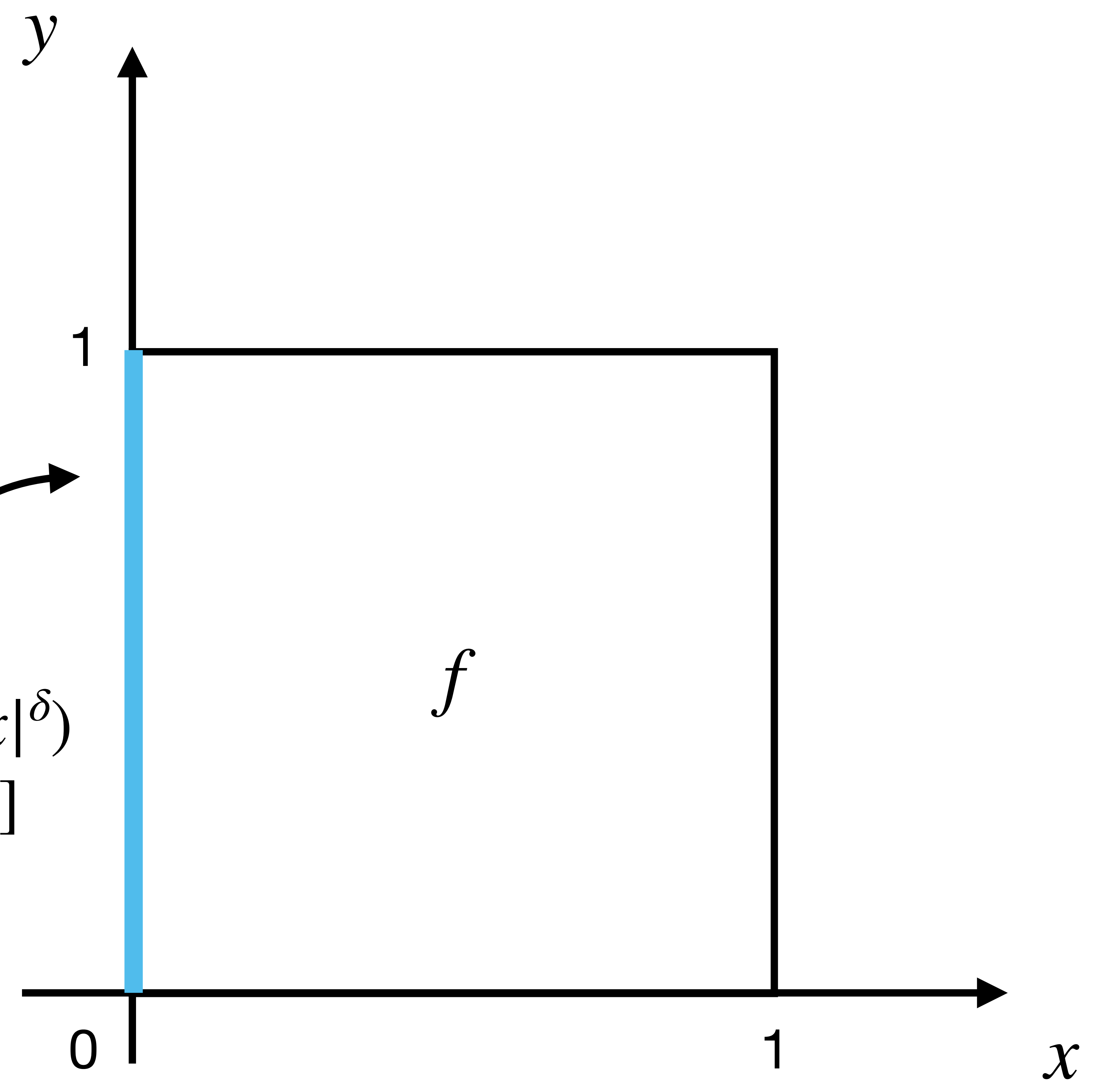
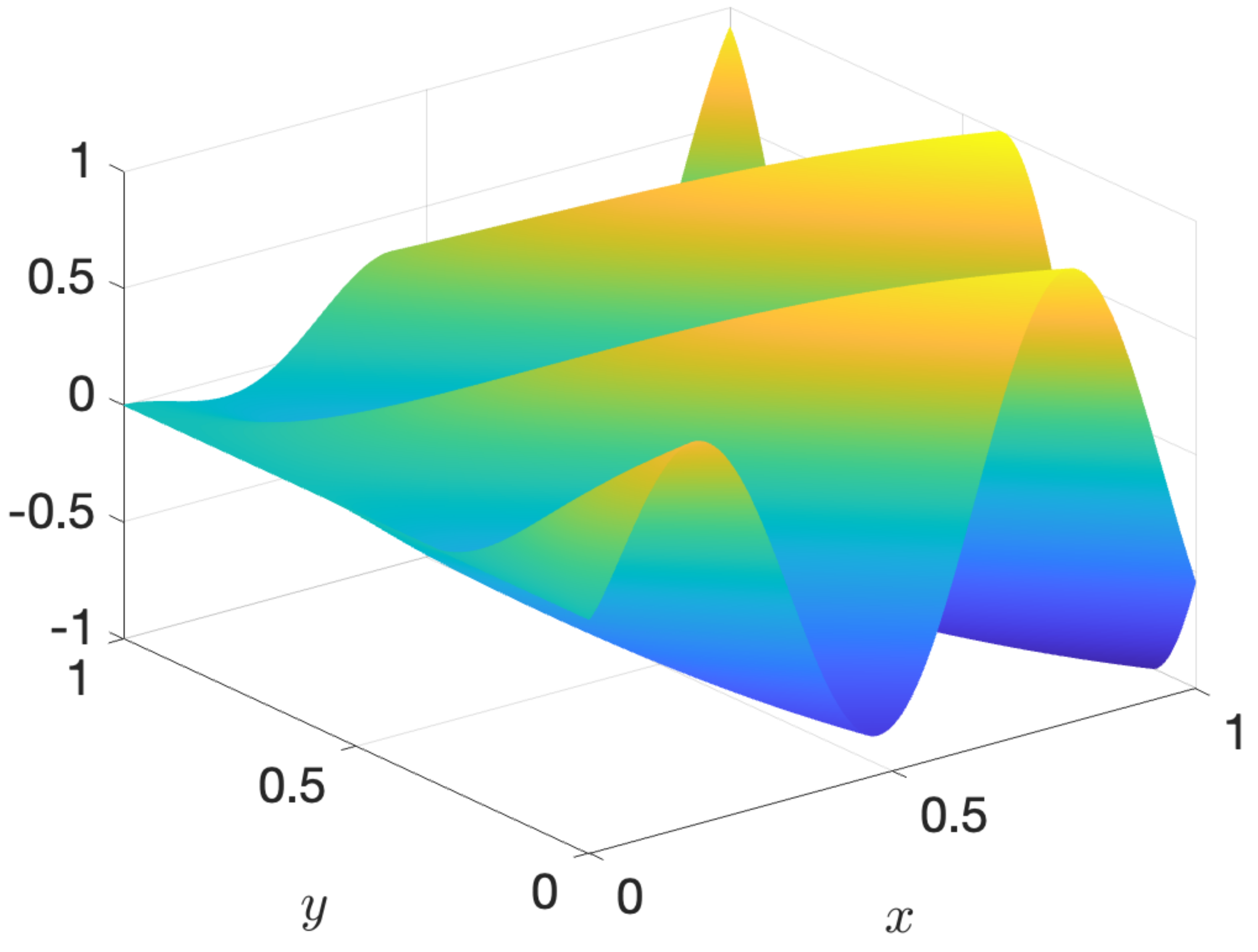


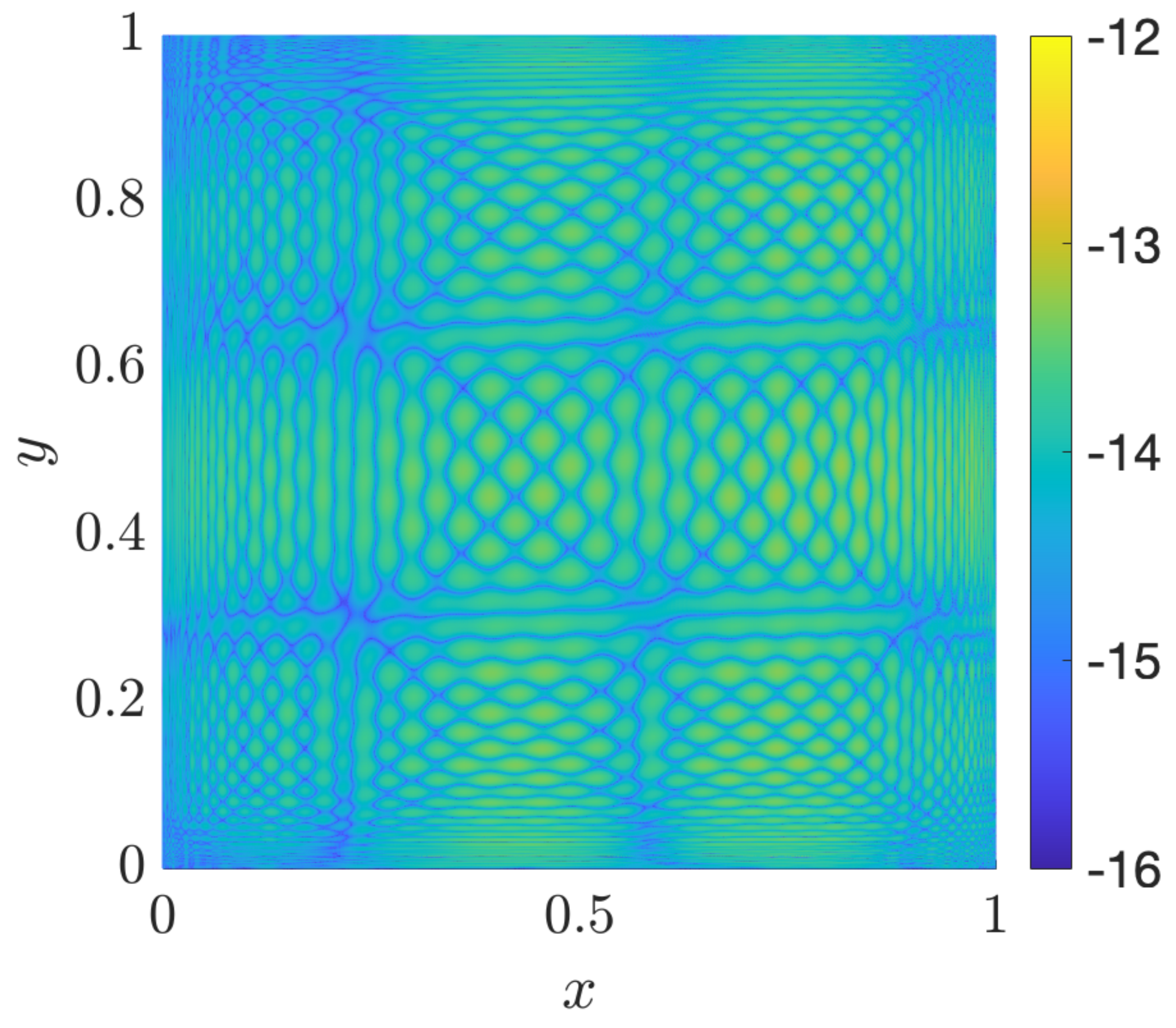
Multivariate case

$$f(x, y) \approx \sum_k \frac{a_k(y) p_k}{x - p_k} + b(x, y)$$

$f = x^{1/4+y} \sin(10(x + y))$



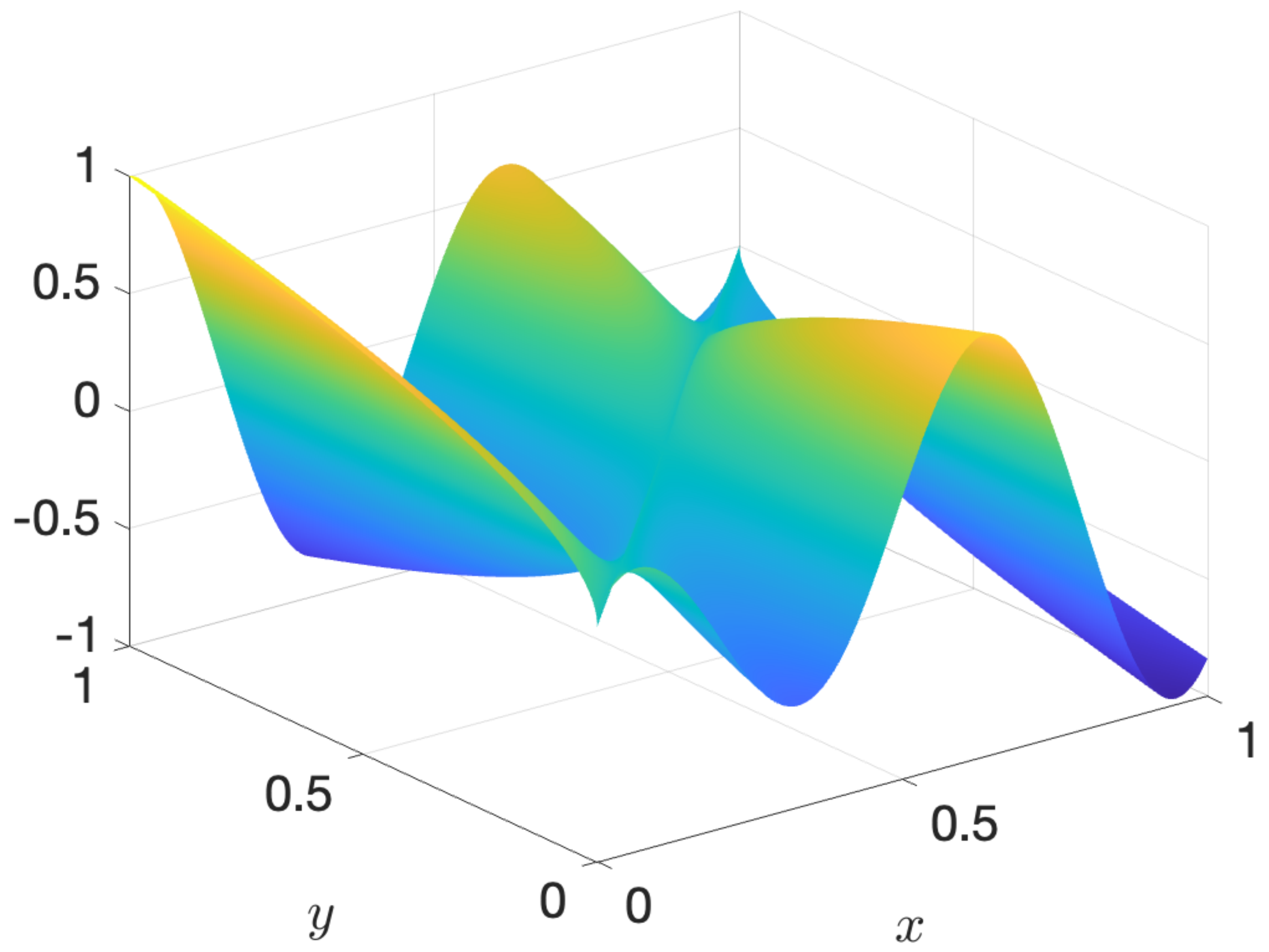
$\log_{10}(\text{error})$



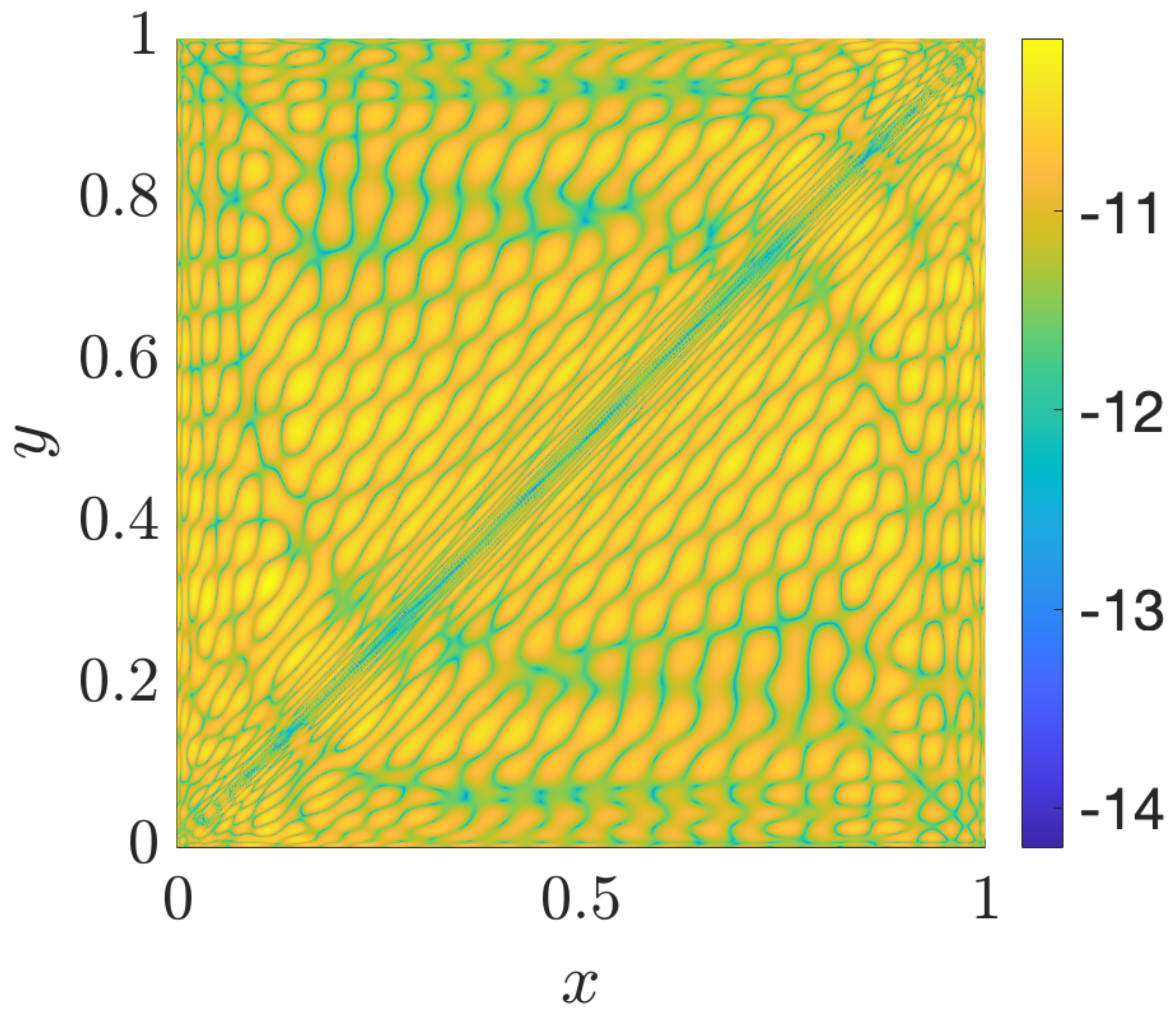
Multivariate case

$$f(x,y) \approx \sum_k \frac{a_k(x+y)p_k}{(y-x) \pm i p_k} + b(x,y)$$

$f = \sqrt{|x-y|}\cos(10x)$



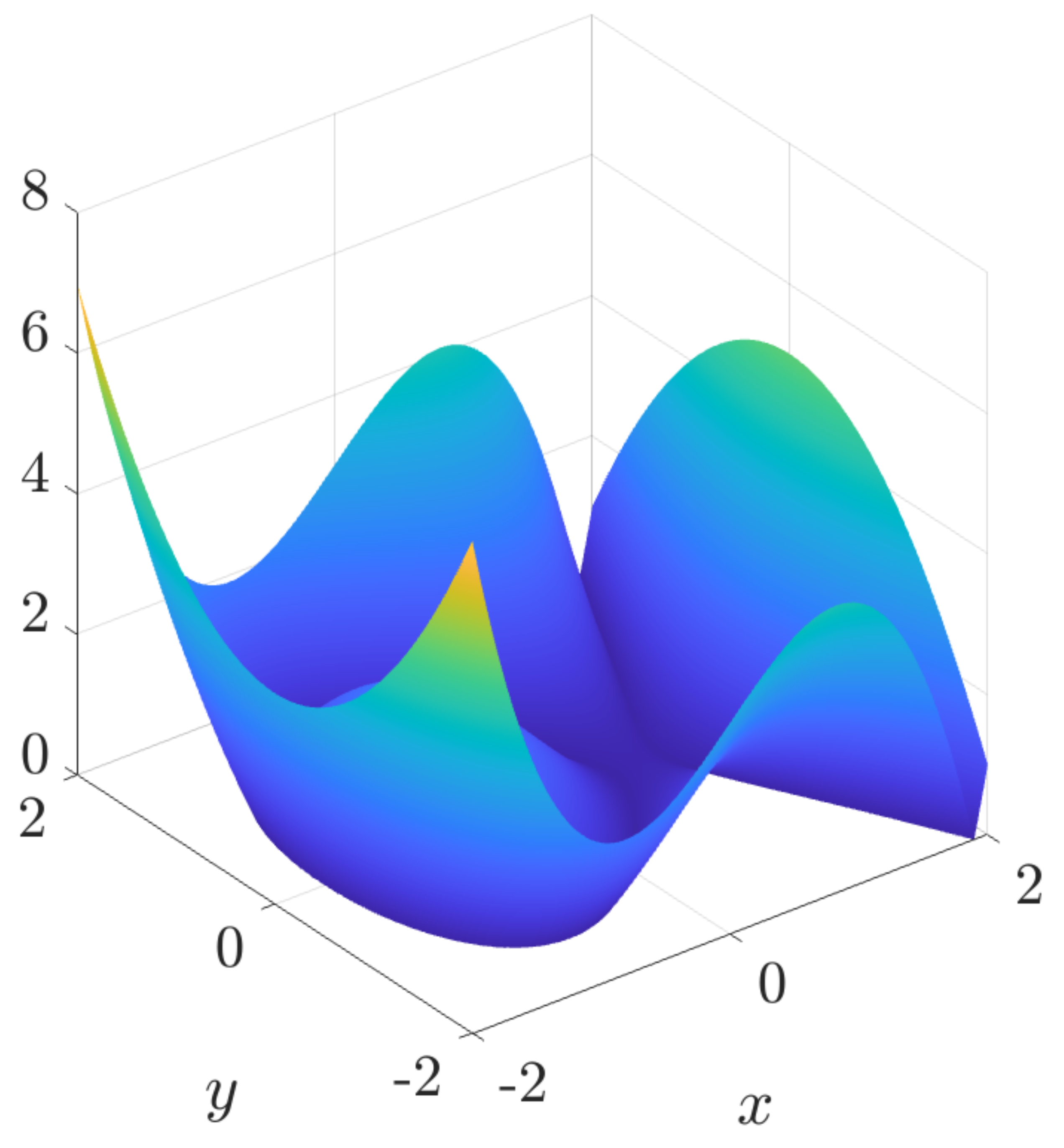
$\log_{10}$(error)



Multivariate case

$$f(x,y) \approx \sum_k \frac{a_k(x,y)p_k}{C(x,y) \pm ip_k} + b(x,y)$$

$f = |x^3 - 2x + 1 - y^2| = |C(x,y)|$



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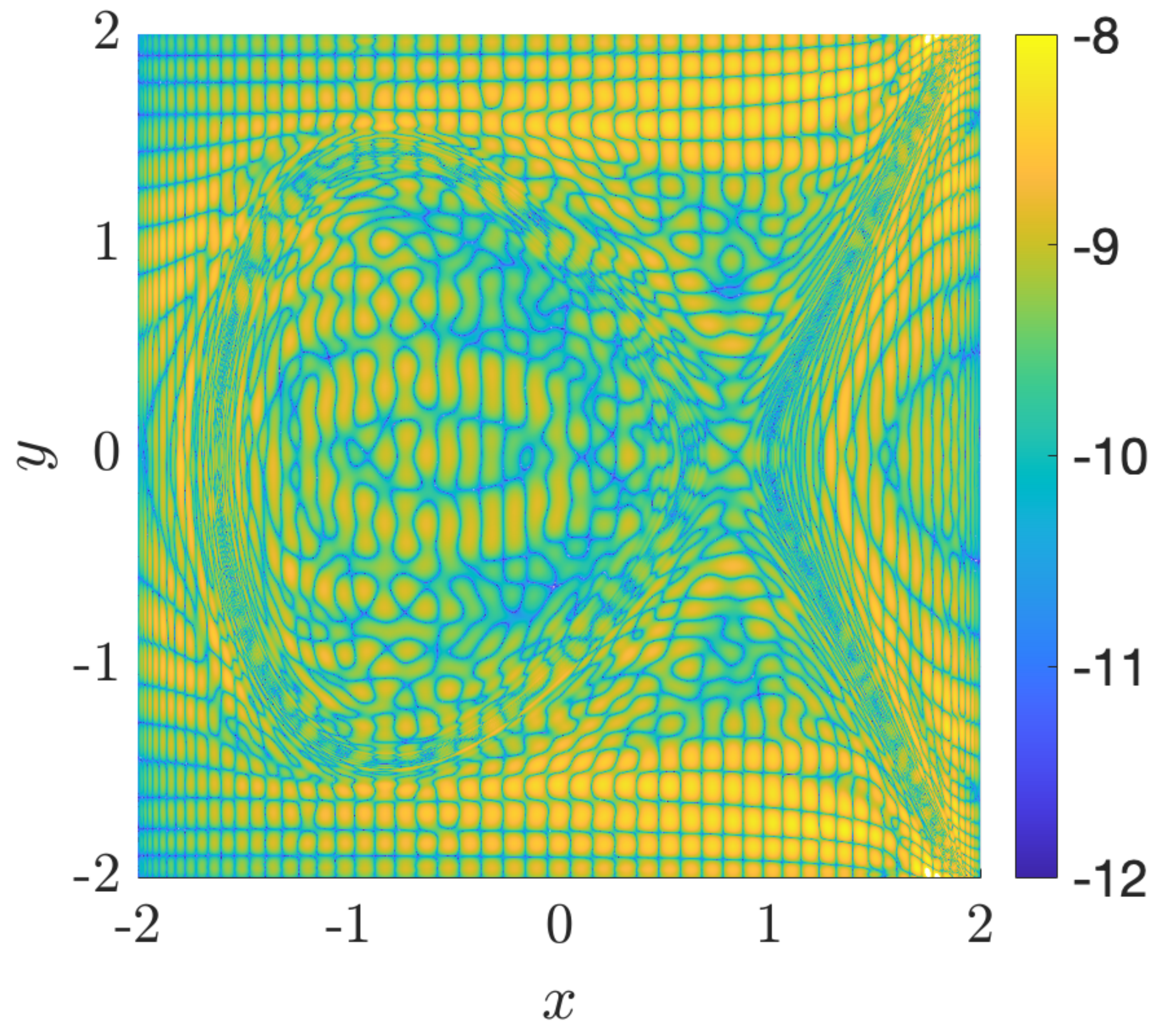
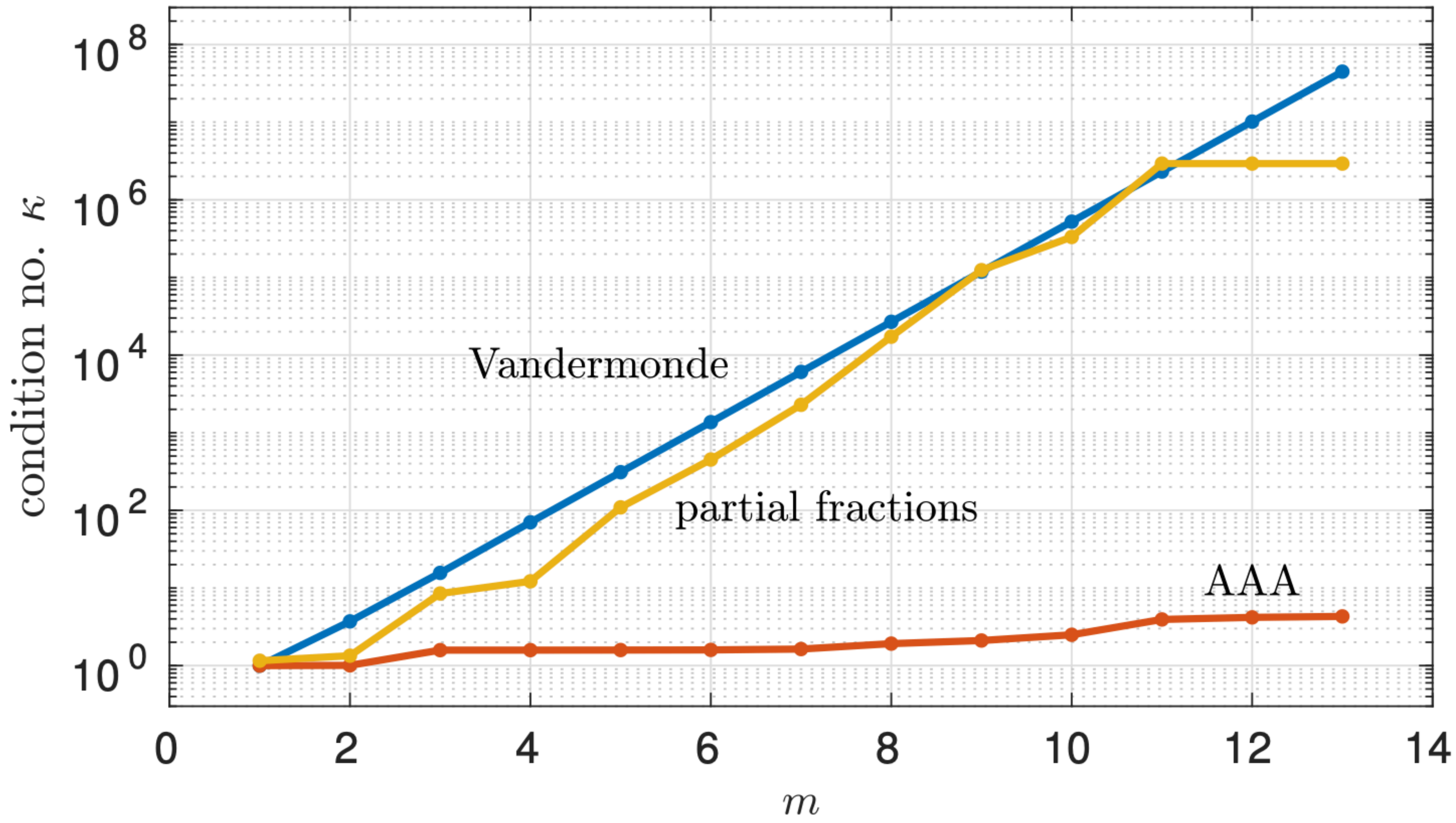
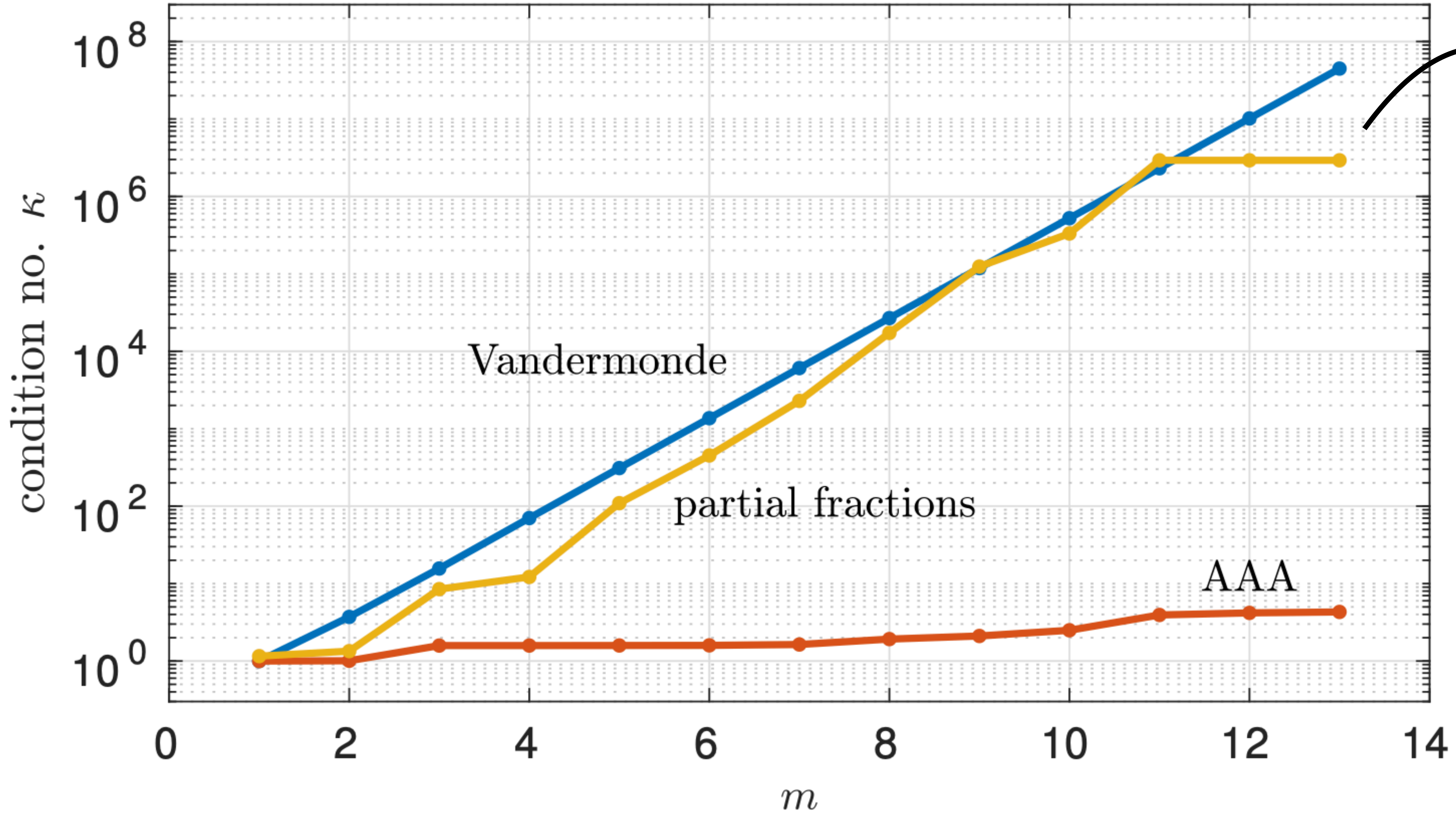


On the partial fractions representation



(Nakatsukasa, Sète and Trefethen, 2018)

On the partial fractions representation



This means that the basis is close to being linearly dependent.
Is this a problem?

(Nakatsukasa, Sète and Trefethen, 2018)

Sensitivity to data

Given sampled data $(b)_i = f(x_i)$, we compute $\hat{f} = \sum_k \hat{c}_k \phi_k$ where

$$\hat{c} = \arg\min_c \|Ac - b\|_2^2 \qquad \text{with } (A)_{i,j} = \phi_j(x_i)$$

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The (absolute) condition number of $b \mapsto \hat{f}$ satisfies

$$\kappa \leq \frac{1}{C} \qquad \text{where } \; C\|v\| \leq \|\{v(x_i)\}_i\|_2, \quad \forall v \in \text{span}(\phi_k)$$

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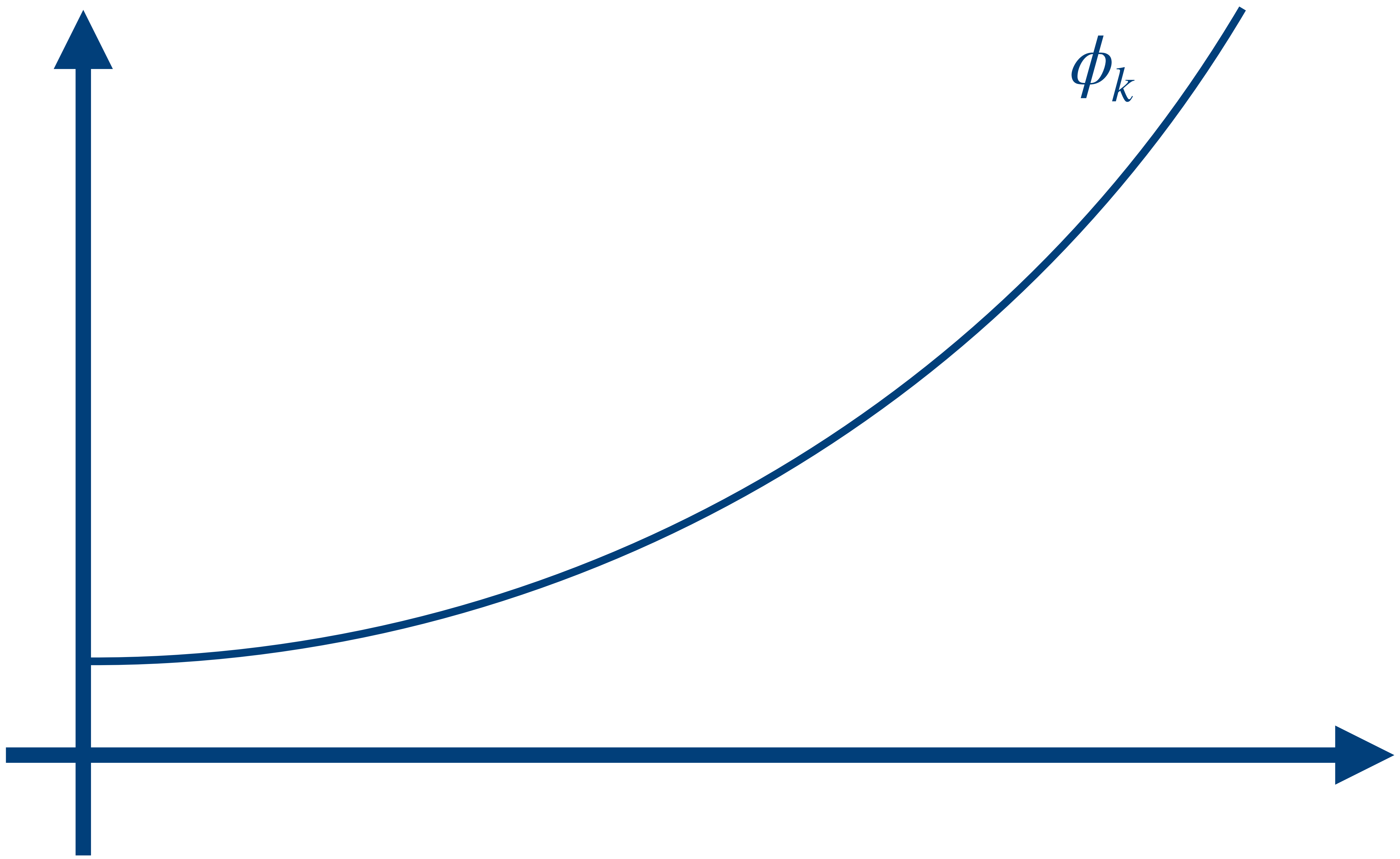
This does <u>NOT</u> depend on (the condition number of) the basis ϕ_k

Numerical approximation space



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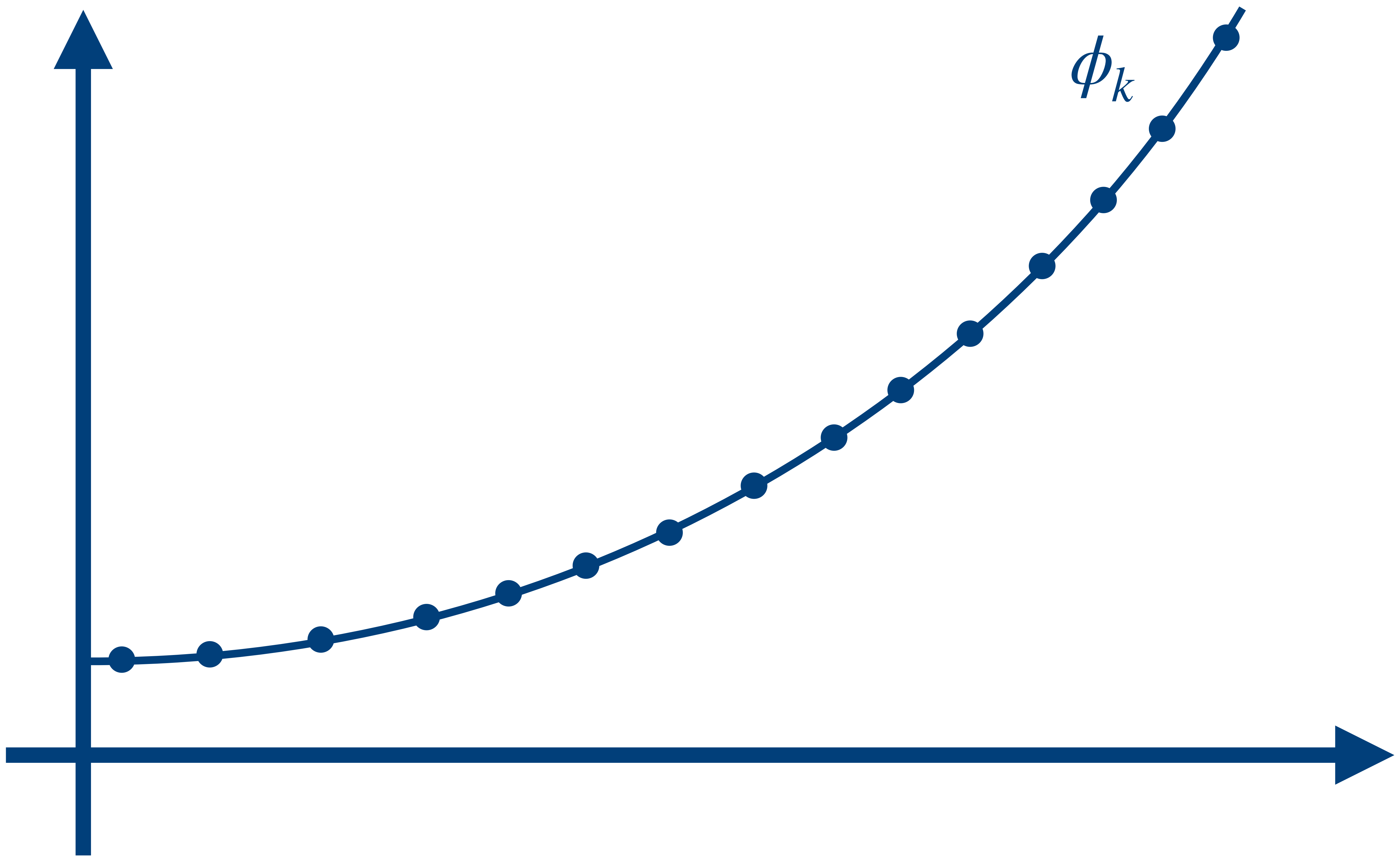
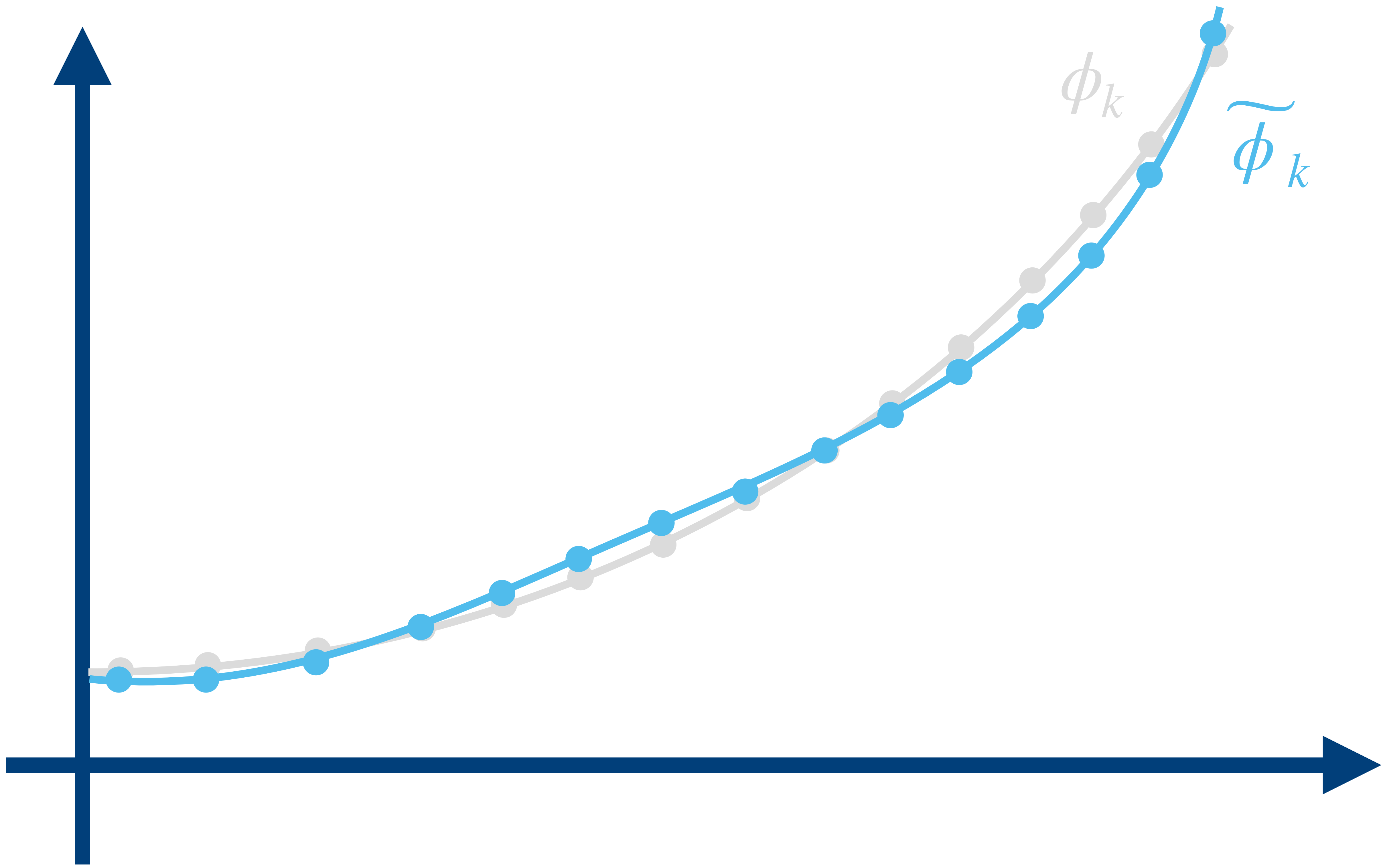


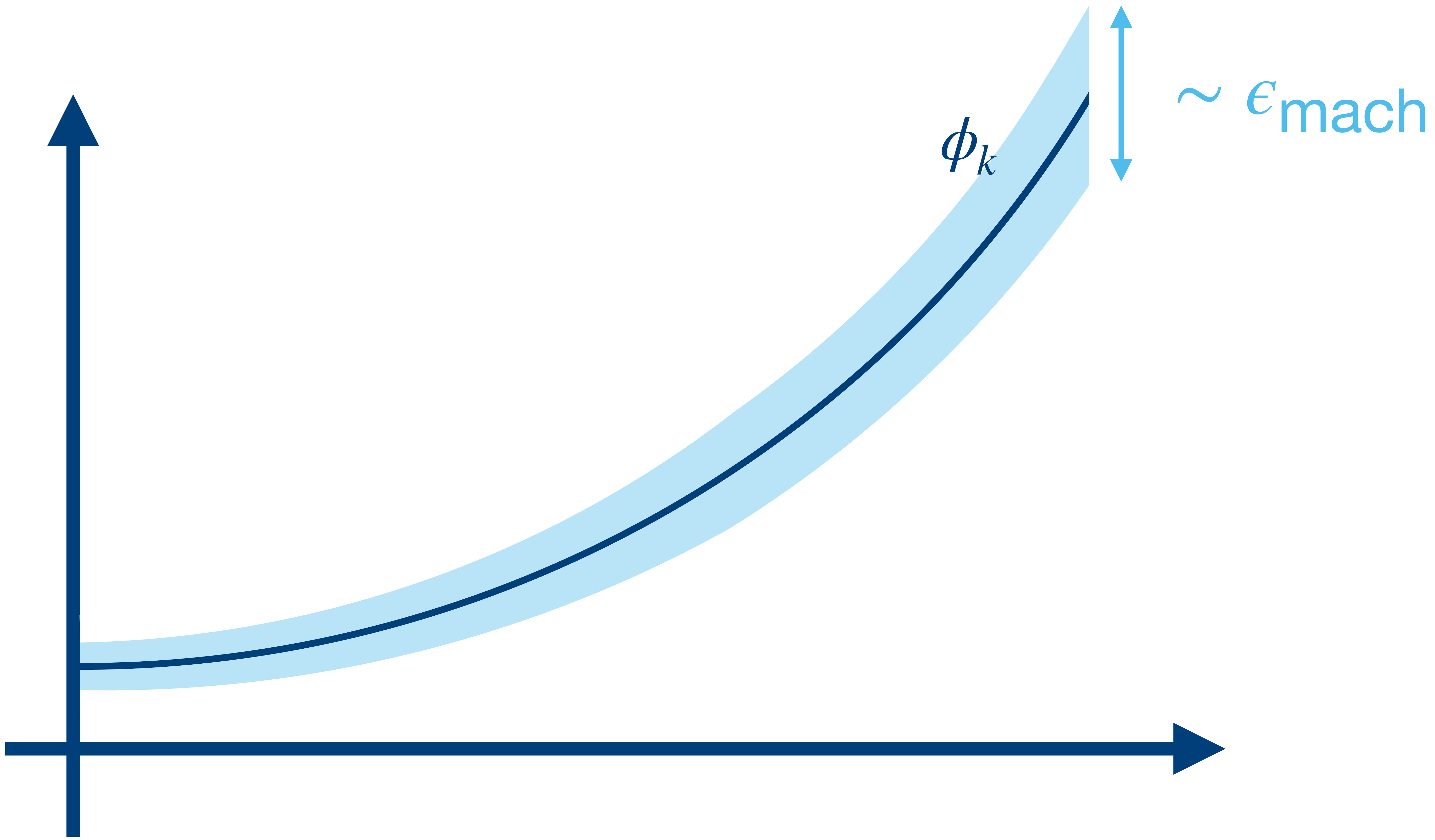
Numerical approximation space

on a computer we work with $\widetilde{\phi}_k$ instead of ϕ_k



Numerical approximation space

we don't know $\widetilde{\phi}_k$ but we know it lies close to ϕ_k



Numerical approximation space

best approximation error in span(ϕ_k) $\qquad \inf_c \left(\|f - \sum_k c_k \phi_k\| \right)$

best approximation error in span$(\widetilde{\phi}_k)$ $\qquad \leq \inf_c \left(\|f - \sum_k c_k \phi_k\| + \epsilon_{\text{mach}}\, B\, \|c\|_2 \right)$

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$\sup_{\|c\|_2 = 1} \| \sum_k c_k \phi_k \|$

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- focus on existence of accurate approximations with small coefficients ($\rightarrow$ frames)

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- the difference is only significant if the basis is "numerically redundant"

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- in practice: use ℓ^2-regularization, mind the scaling of ϕ_k

Numerical approximation space

(Adcock and Huybrechs, 2019/2020), (H. and Huybrechs, 2025)

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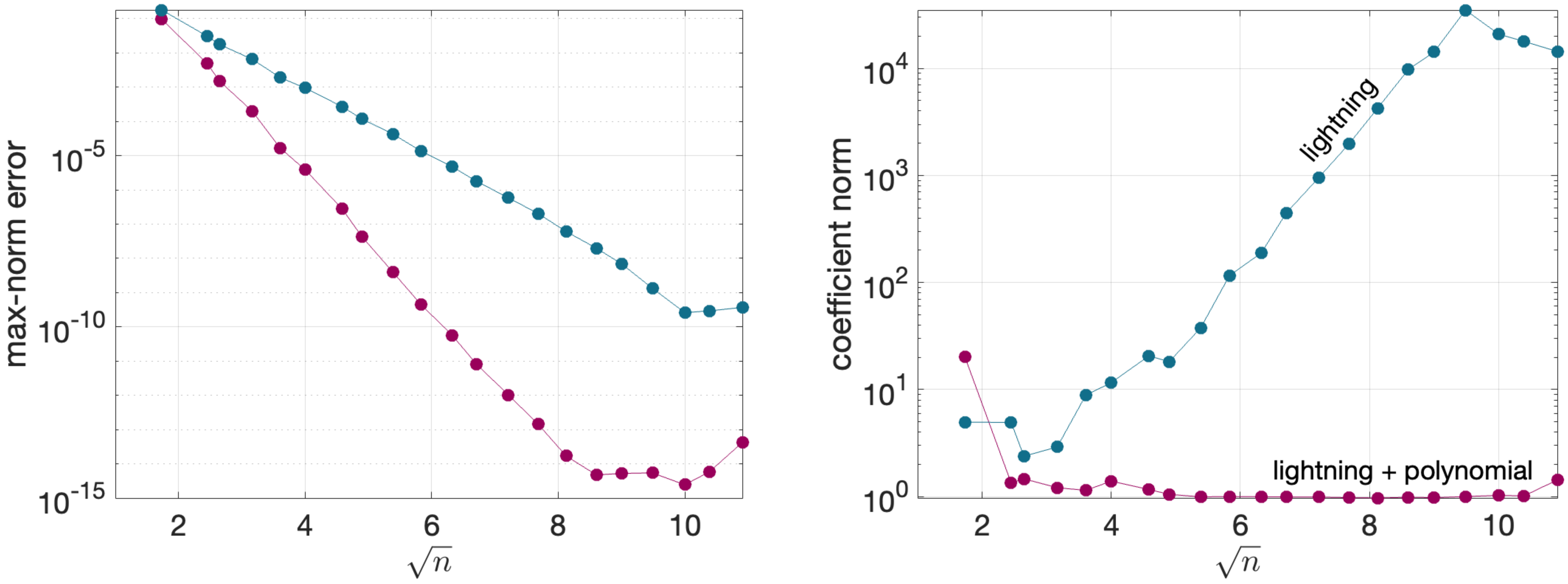
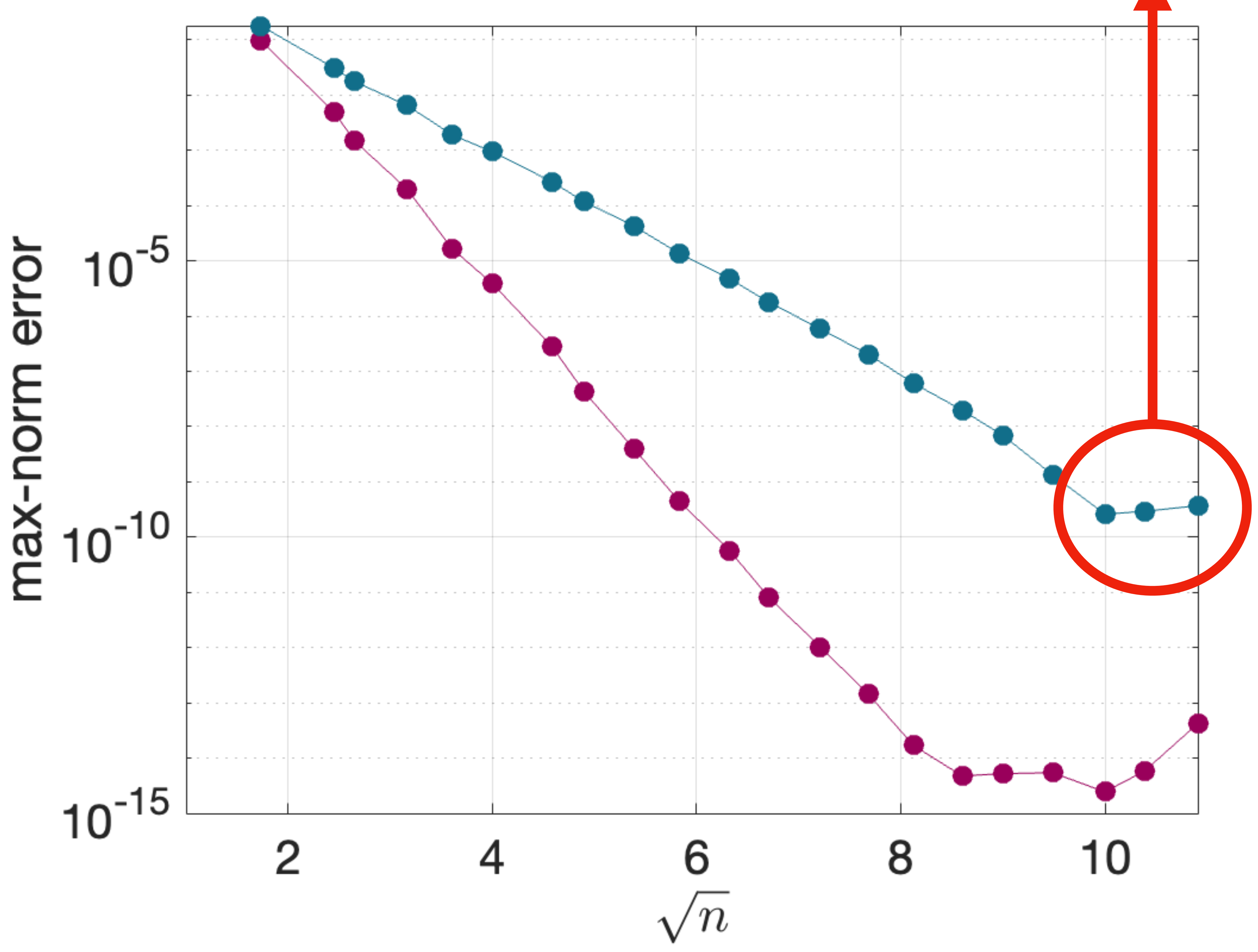
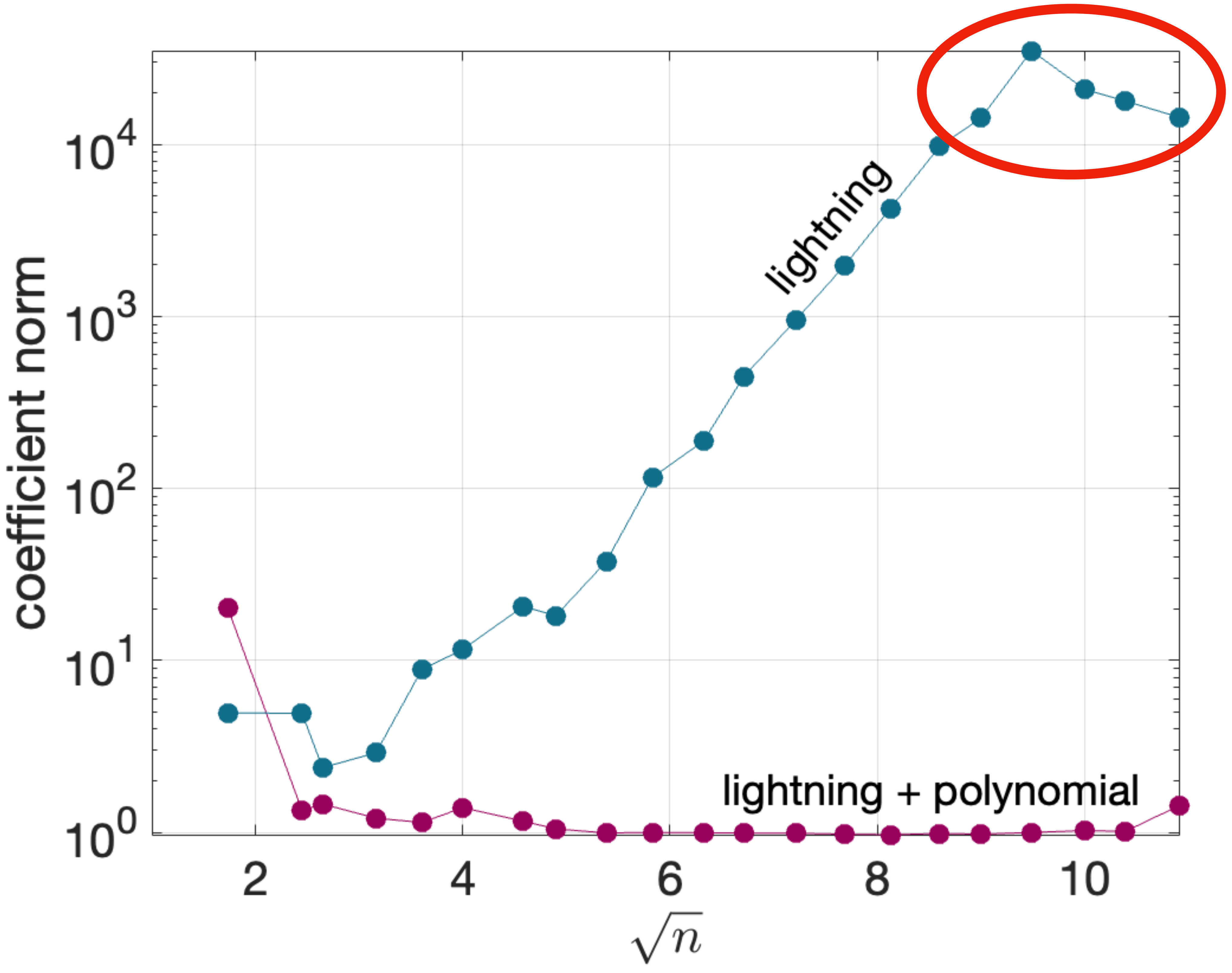


(H., Huybrechs and Trefethen, 2023)

Numerical approximation space

error stagnates in the numerical
approximation space





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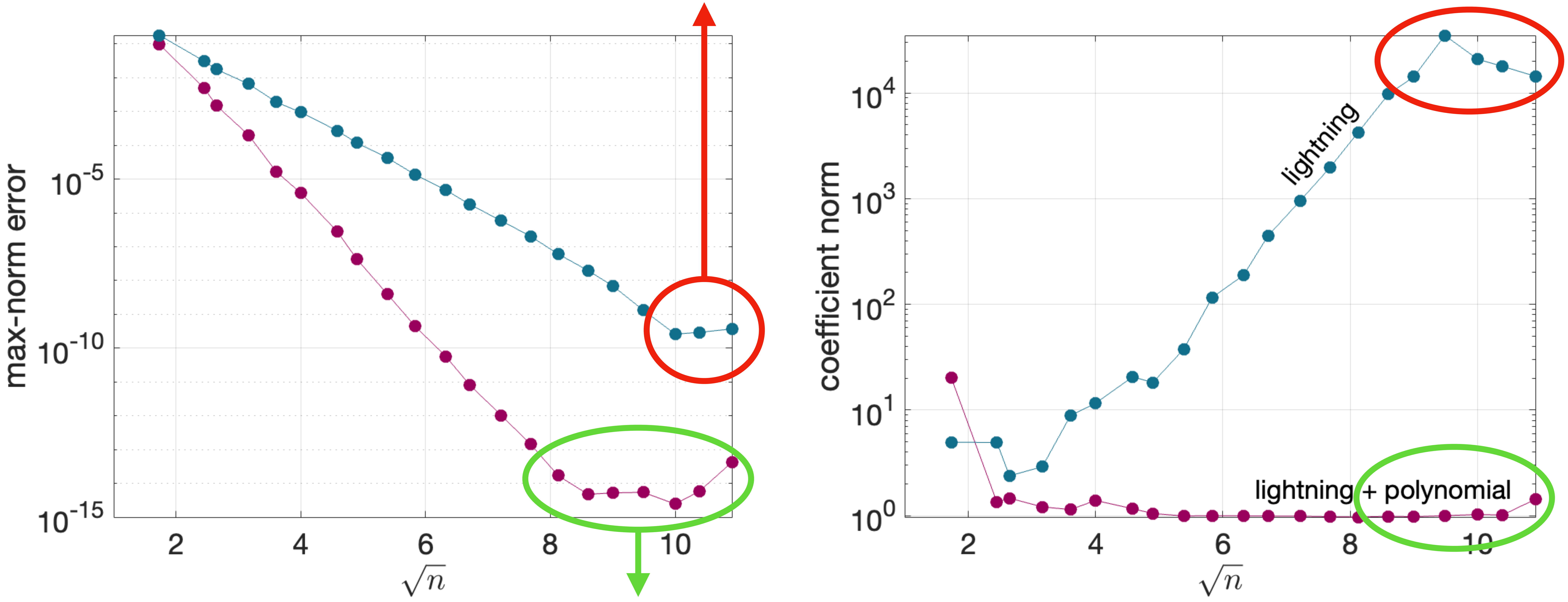

convergence in the numerical
approximation space

(H., Huybrechs and Trefethen, 2023)

Conclusions

Given the <u>locations</u> of the singularities of a function, we can construct root-exponentially converging rational approximations via least squares fitting

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- many mysteries have been solved in 1D
- lots of exploring to do in higher dimensions
- the lightning basis is “ill-conditioned” yet accurate approximations exist in the numerical approximation space → no need to panic ☺