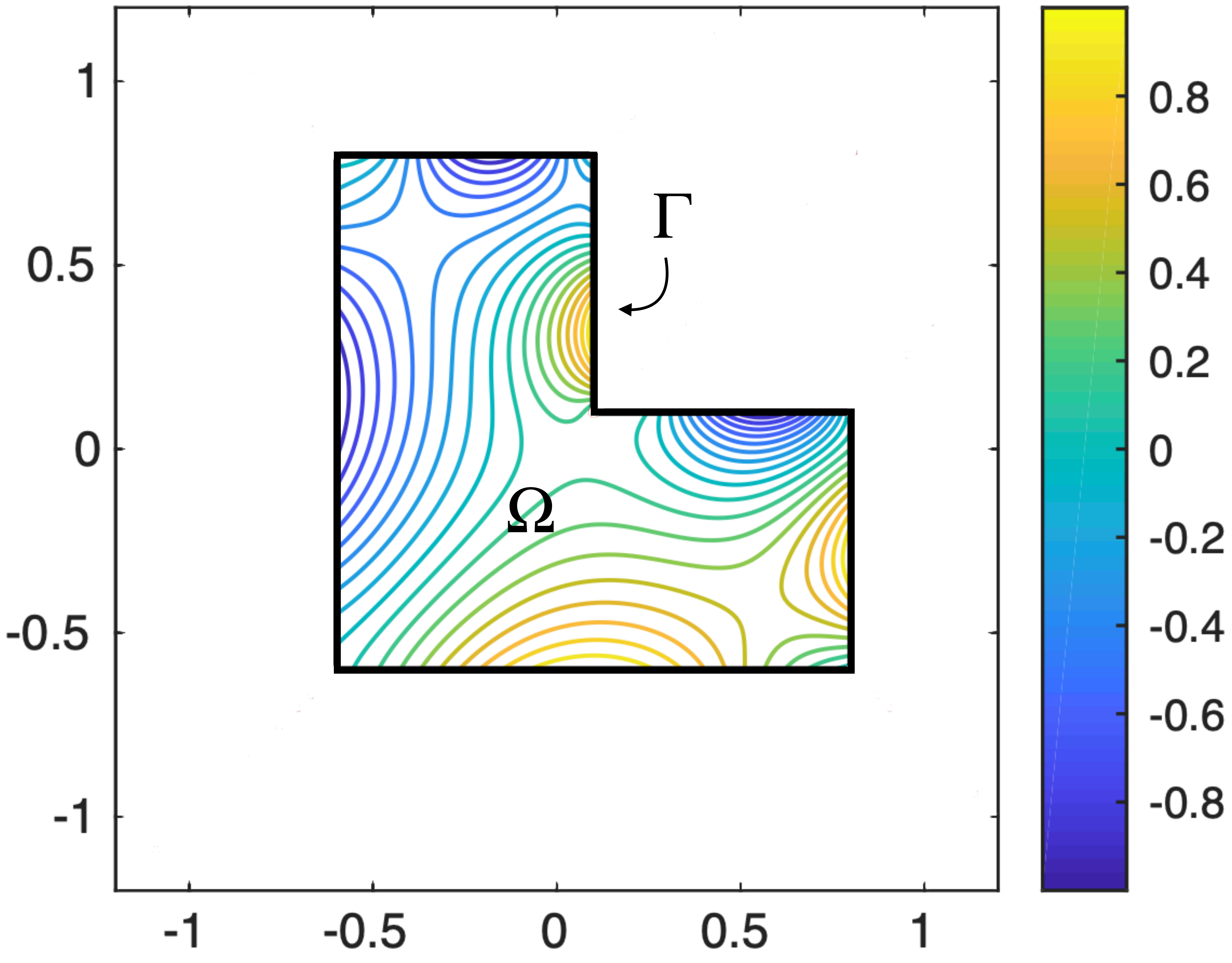
KU LEUVEN

Sampling for function approximation in non-orthogonal bases

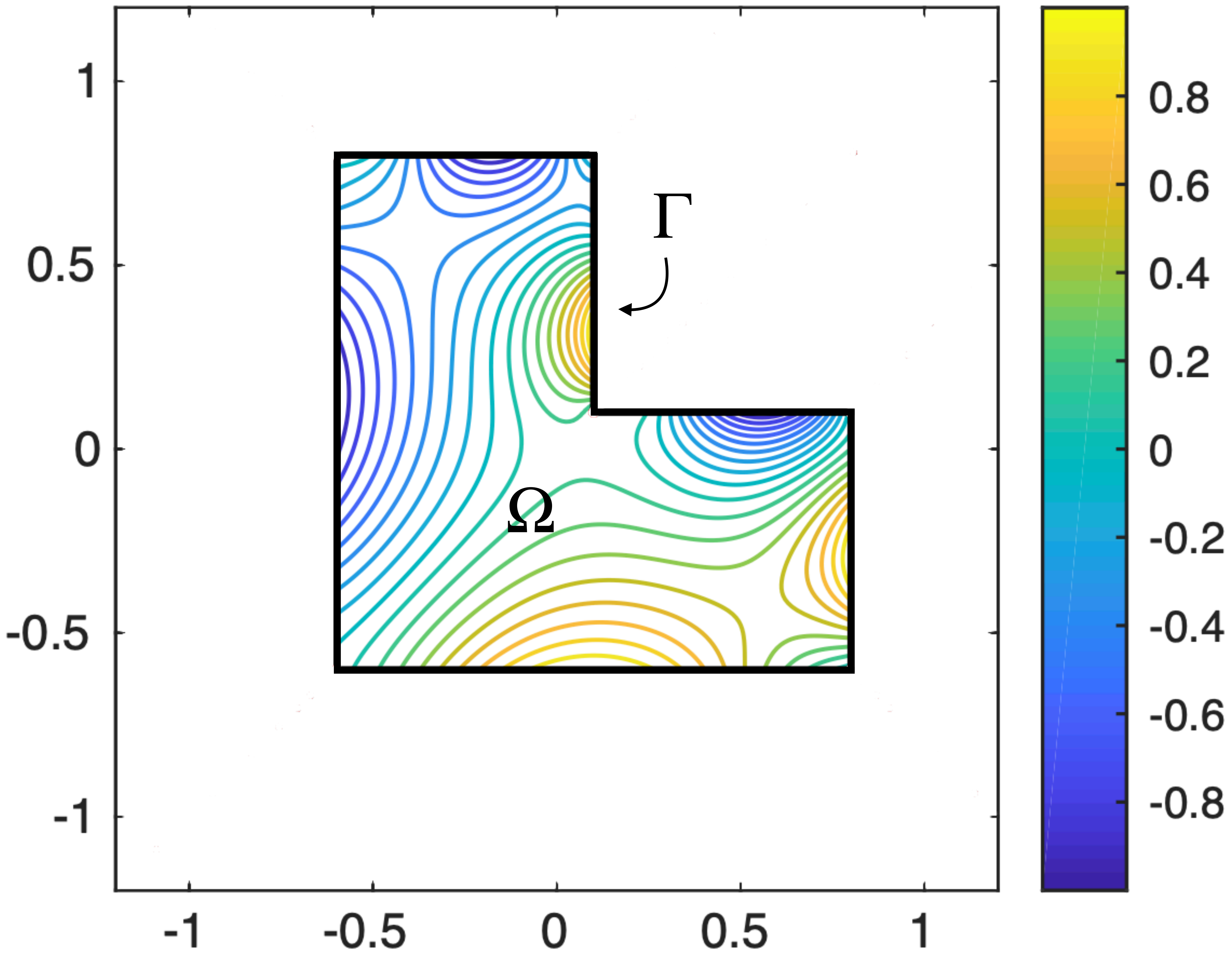
Astrid Herremans
joint work with Daan Huybrechs, Ben Adcock and Lloyd Nick Trefethen

NUMA fwo



$$\Delta u(z) = 0,\ z \in \Omega \qquad u(z) = h(z),\ z \in \Gamma$$

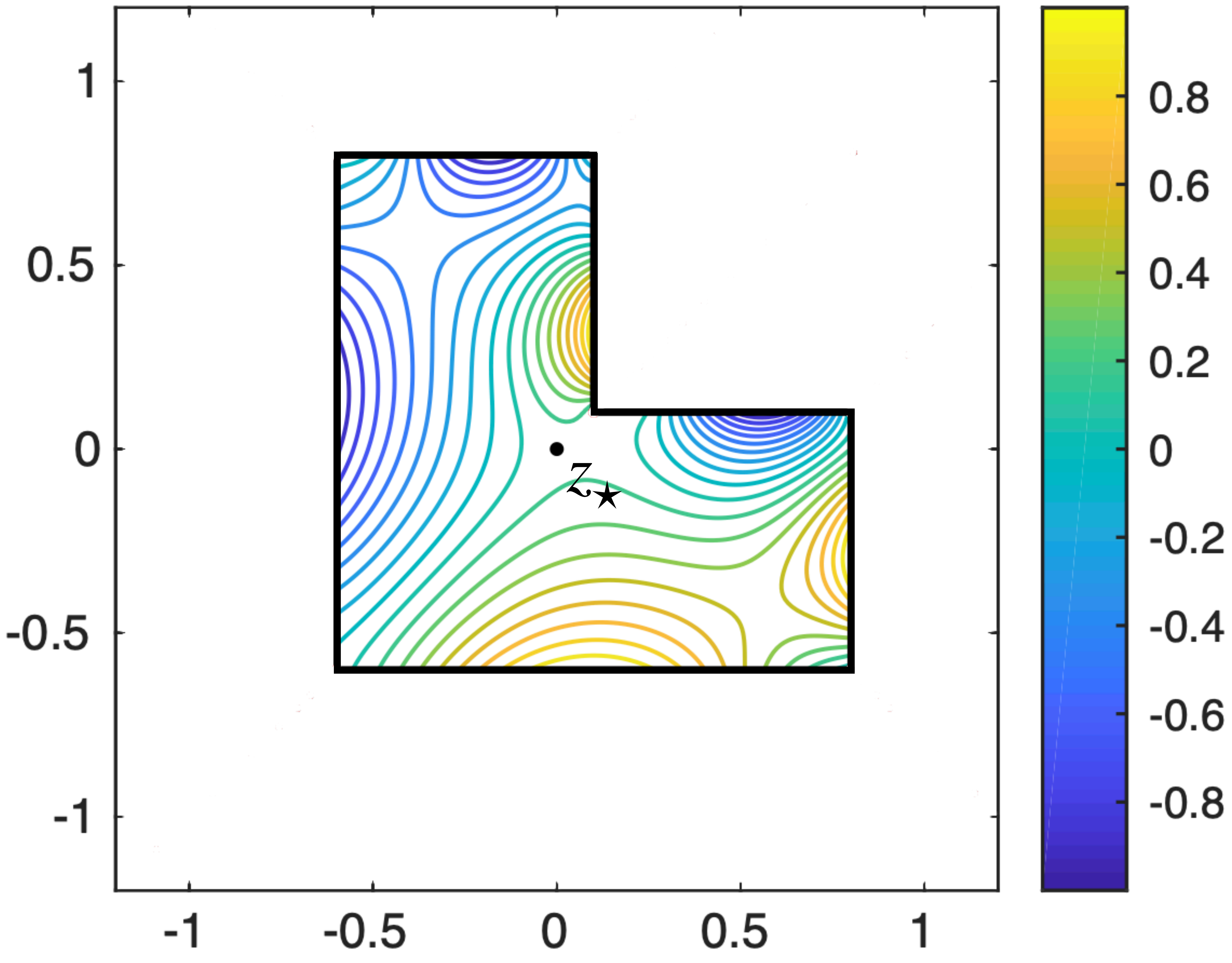
(Gopal and Trefethen, 2019)



automatically satisfied
for harmonic functions

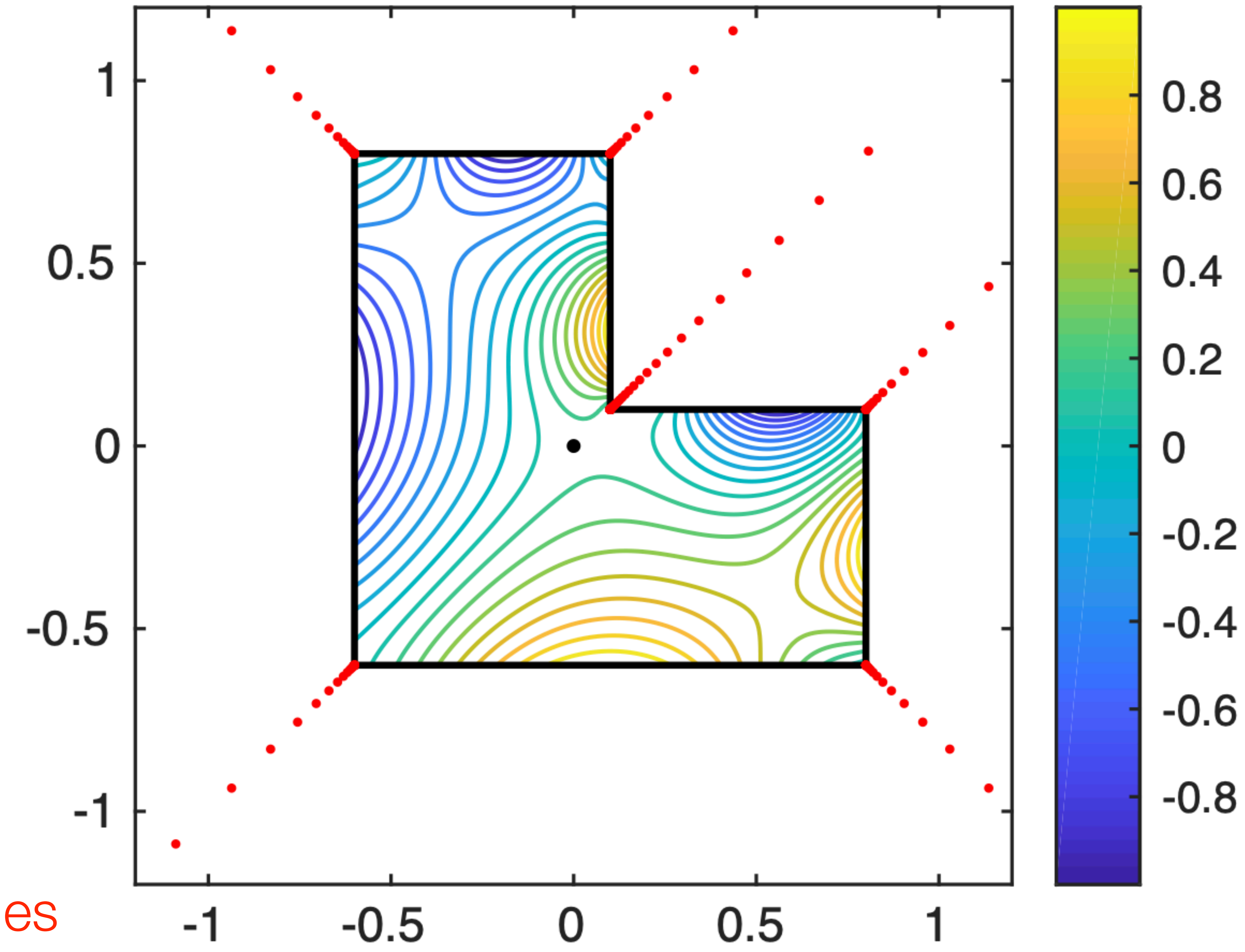
$$\Delta u(z) = 0, \; z \in \Omega \qquad u(z) = h(z), \; z \in \Gamma$$

(Gopal and Trefethen, 2019)



$$u(z) \approx \Re\left(\sum_j c_j (z - z_\star)^j\right)$$

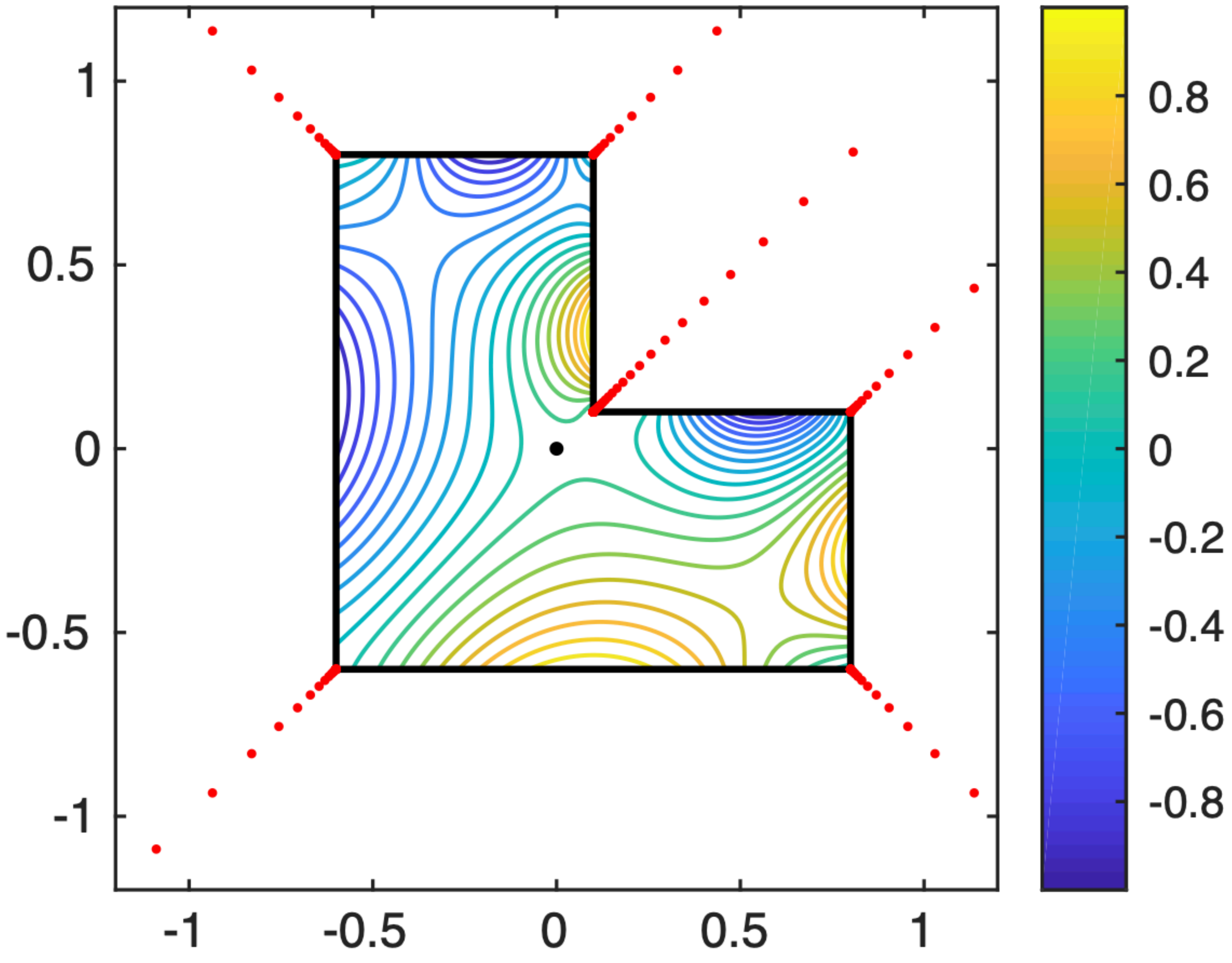
(Gopal and Trefethen, 2019)



"lightning" poles

$$u(z) \approx \Re\left(\sum_j \frac{c_j^{(1)}}{z - z_j} + \sum_j c_j^{(2)}(z - z_\star)^j\right)$$

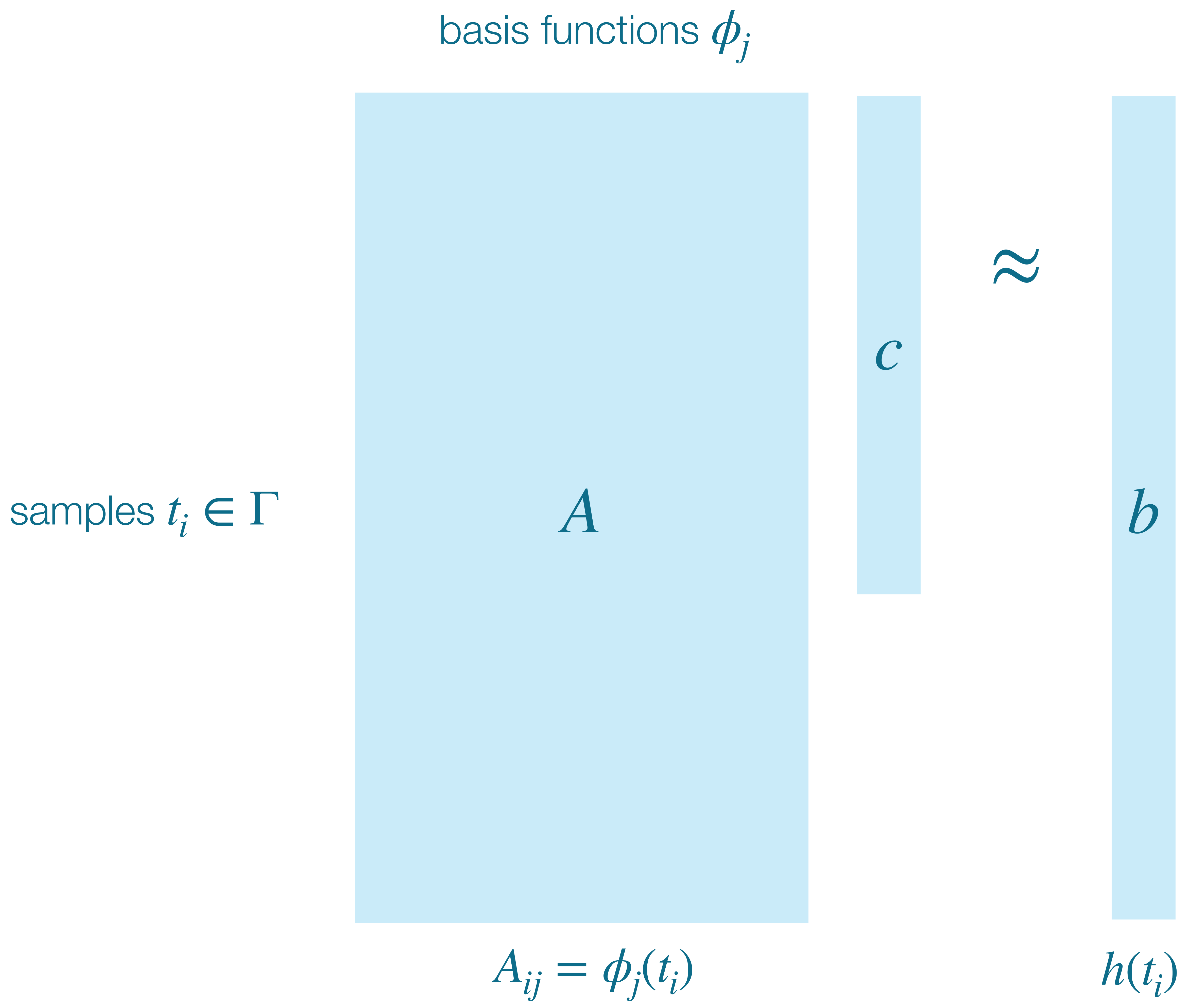
(Gopal and Trefethen, 2019)

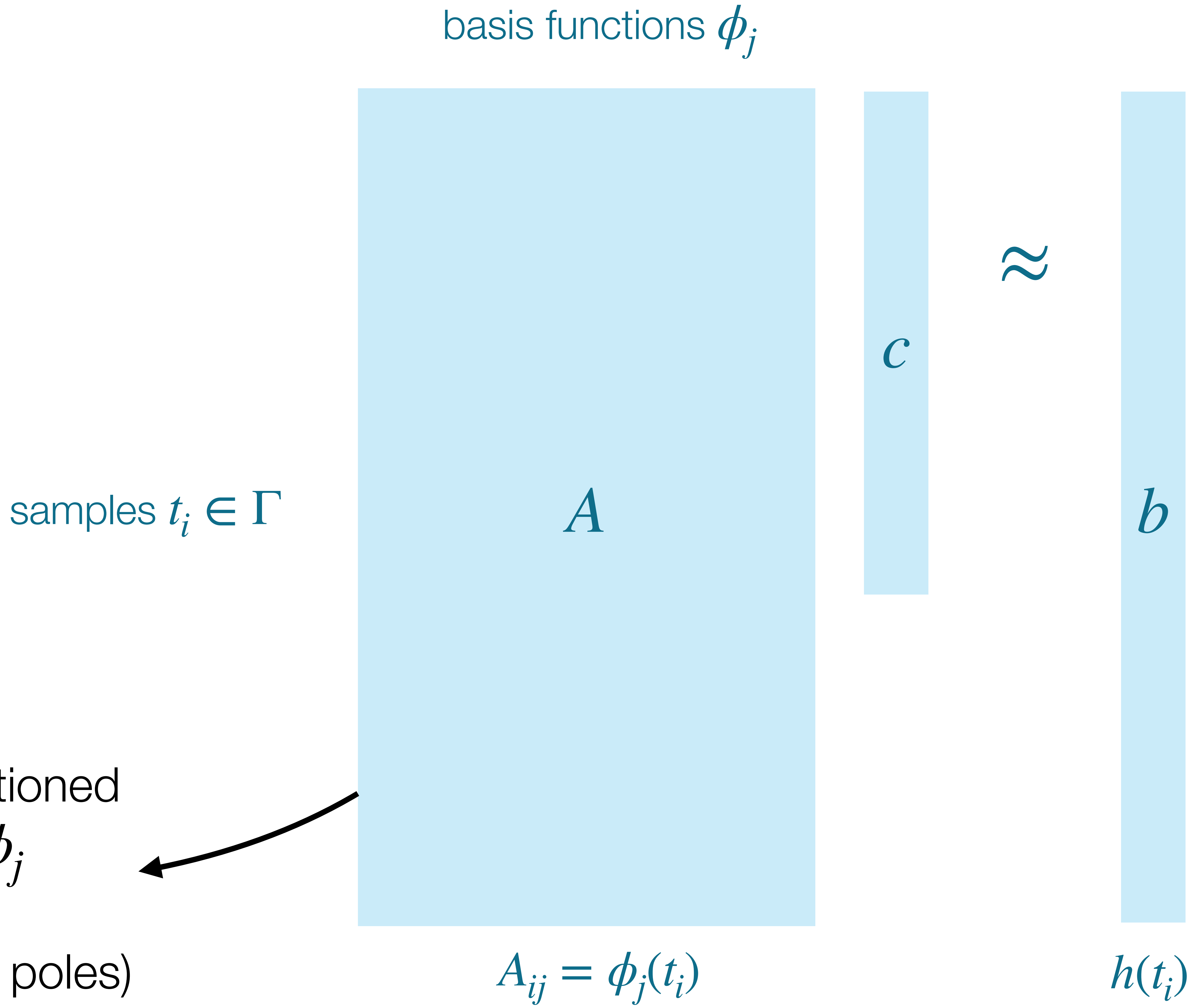


$$u(z) \approx \Re\left(\sum_j \frac{c_j^{(1)}}{z - z_j} + \sum_j c_j^{(2)} (z - z_\star)^j \right)$$

find coefficients $c_j^{(1)}$ and $c_j^{(2)}$ via $u(z) = h(z),\ z \in \Gamma$

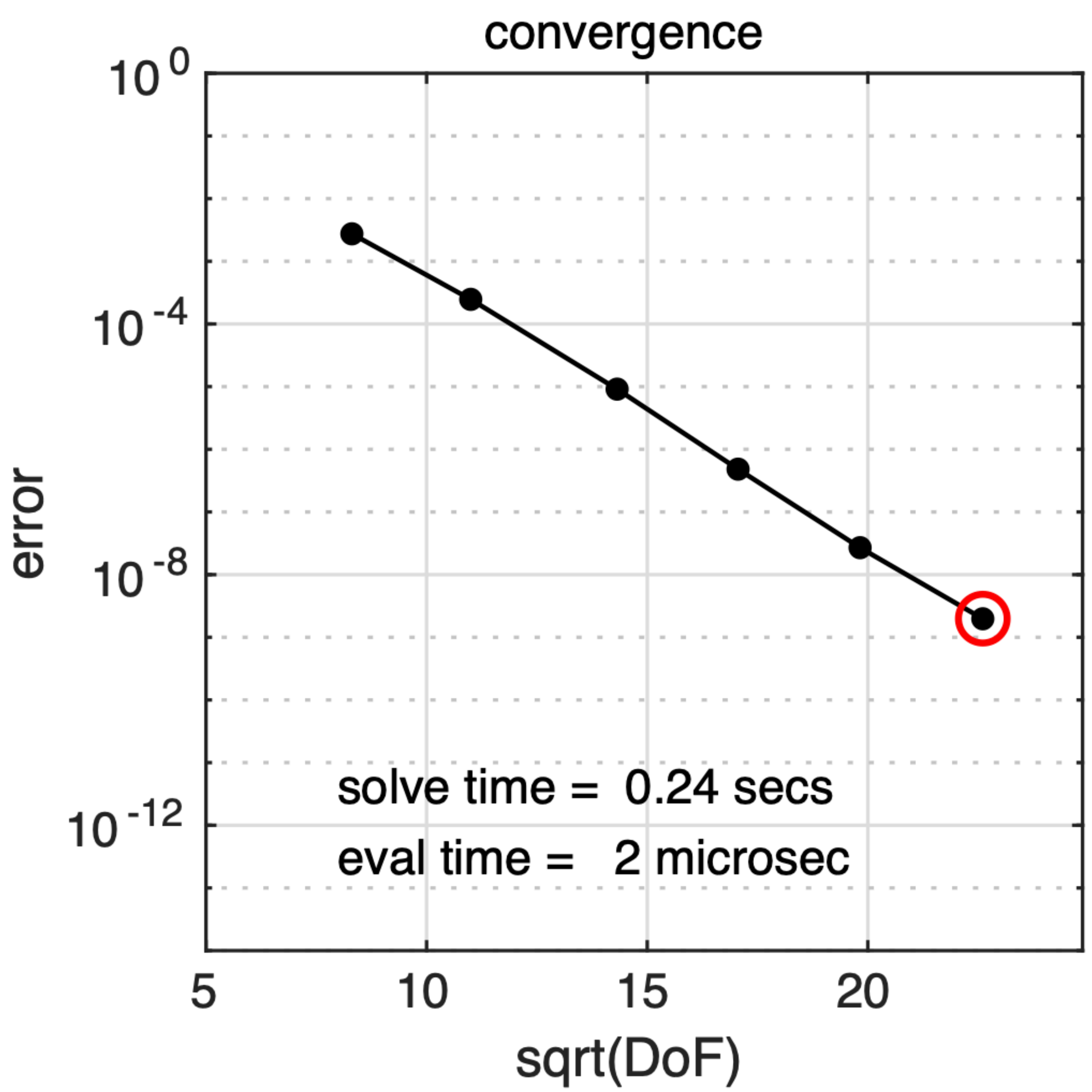
(Gopal and Trefethen, 2019)

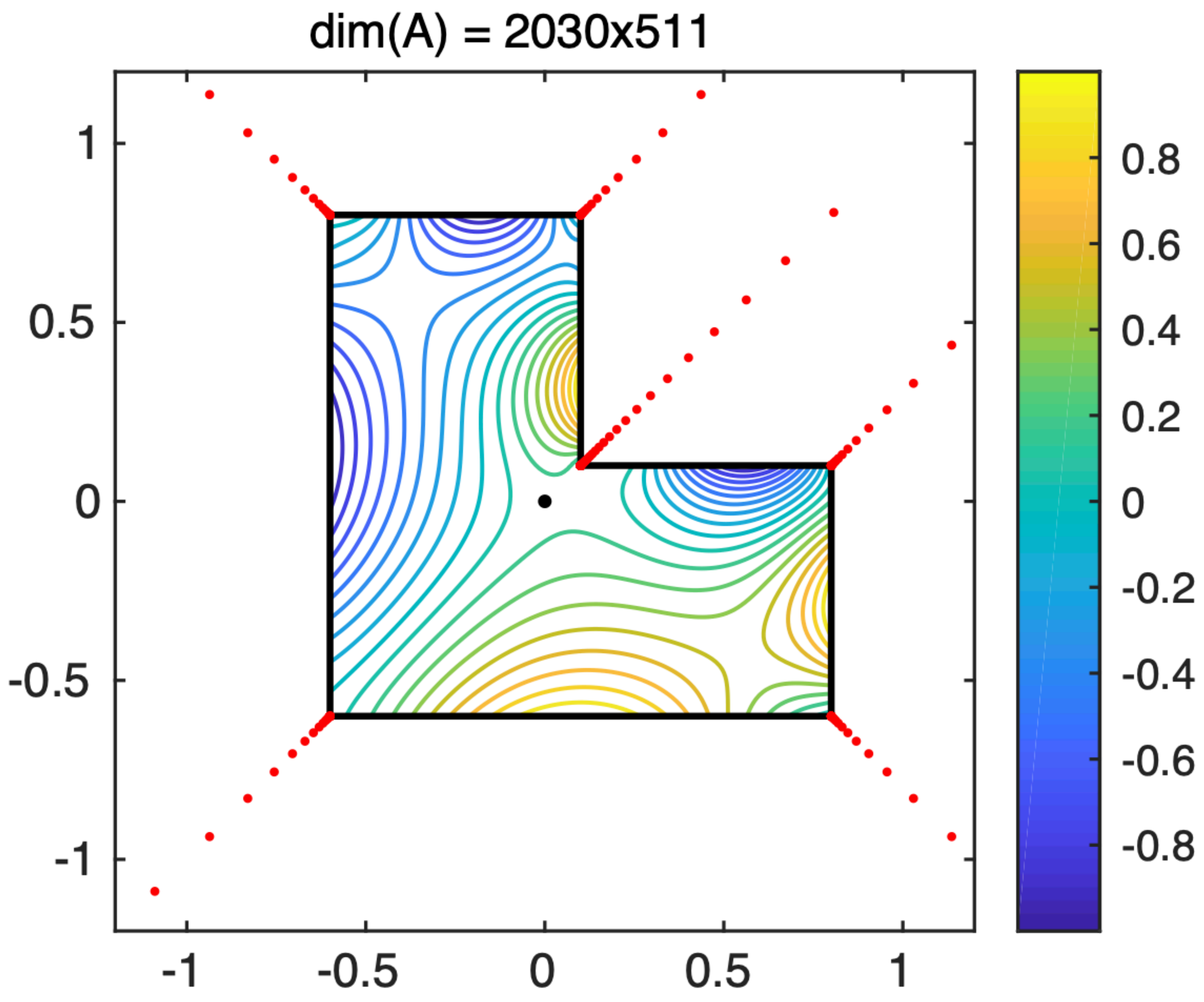

basis functions ϕ_j
samples $t_i \in \Gamma$
A
c
$\approx$
b
$A_{ij} = \phi_j(t_i)$
$h(t_i)$



can be arbitrarily ill-conditioned
for non-orthogonal ϕ_j

(columns = polynomials + poles)

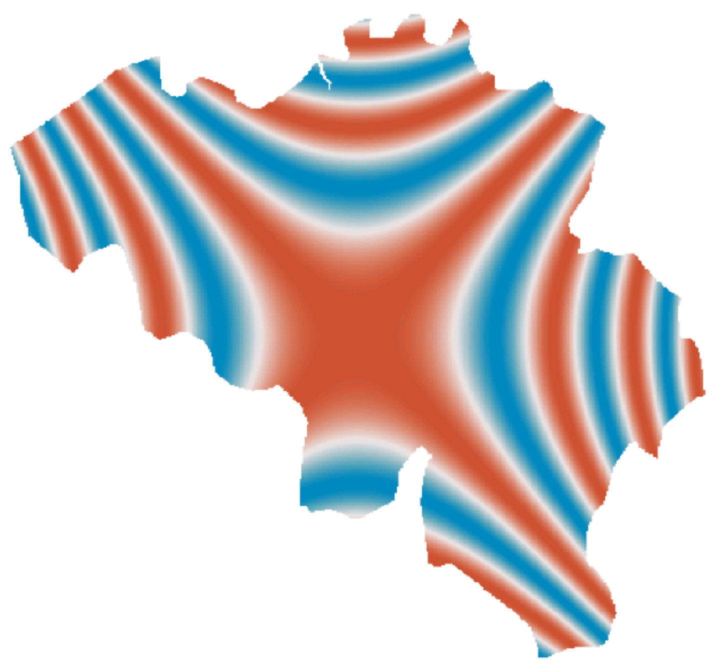




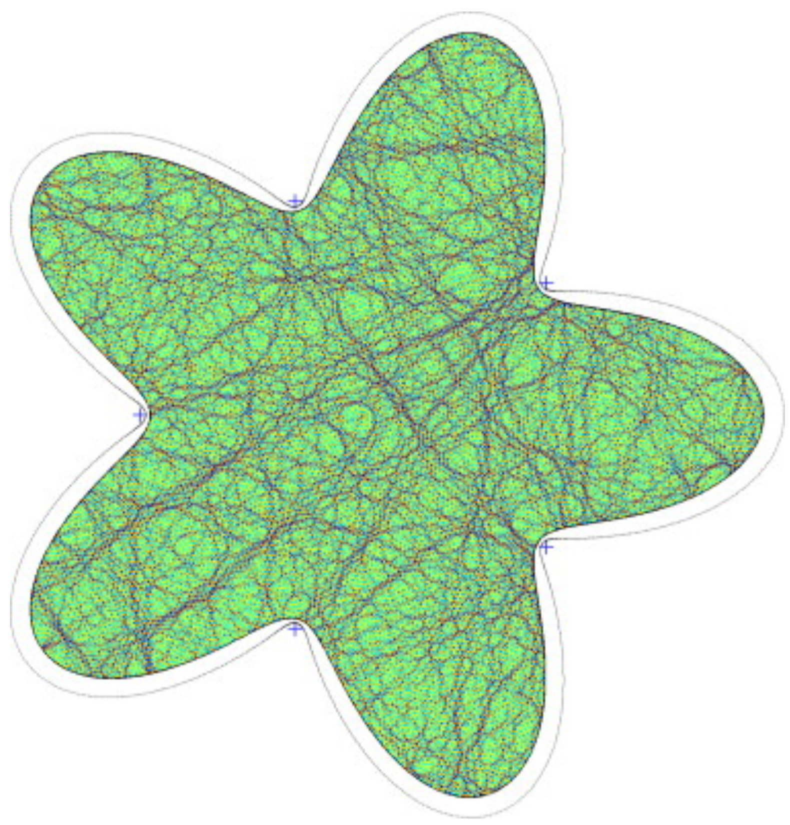
$$\mathrm{cond}(A) \gtrsim 10^{16}$$

(Gopal and Trefethen, 2019)

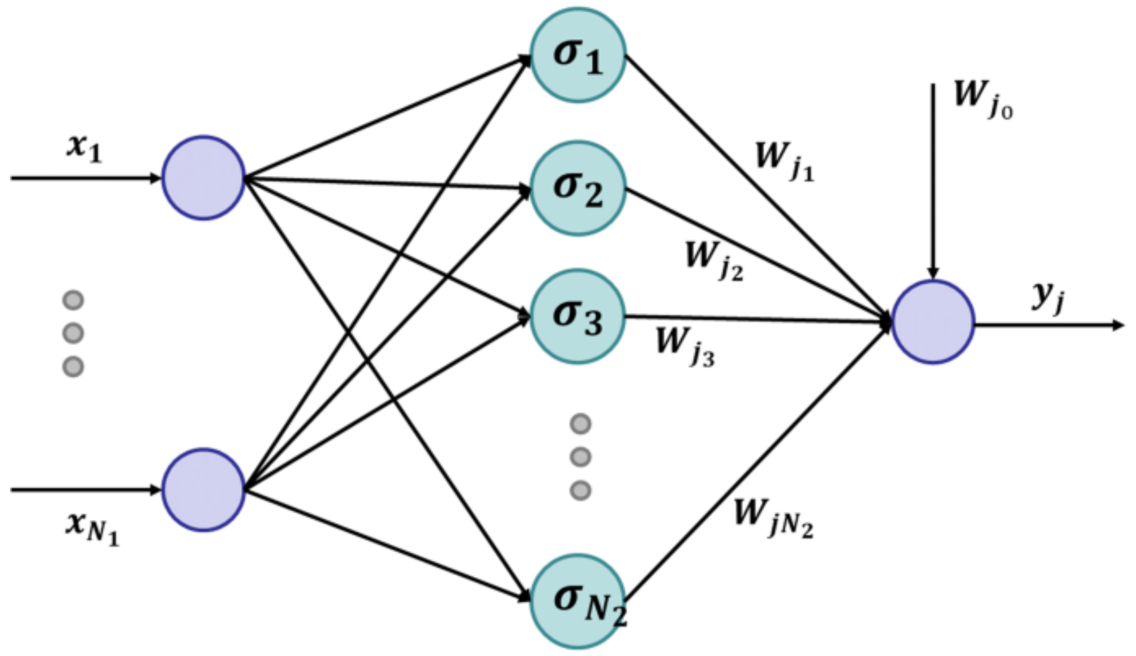
Non-orthogonal bases



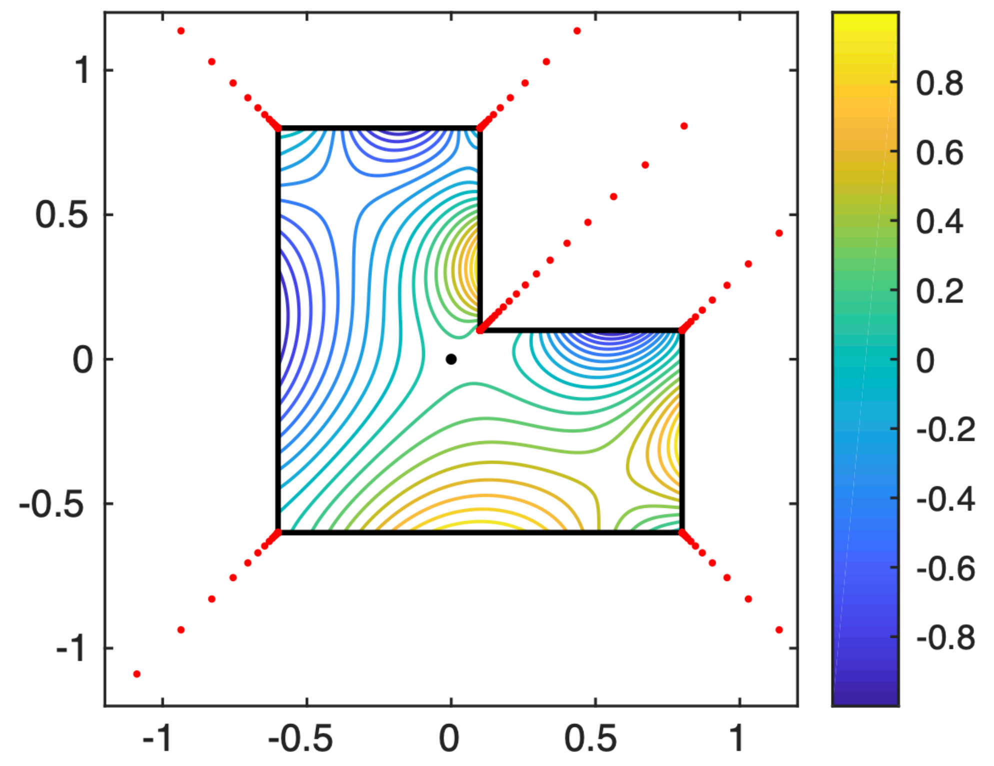
approximating on
irregular domains



Trefftz methods
for solving PDEs



adaptive basis viewpoint
of neural networks



incorporating expert
knowledge on f

(Matthysen and Huybrechs, 2018), (Barnett and Betcke 2008), (Gopal and Trefethen, 2019)

- Approximation theory in finite precision
- An intuitive randomised sampling strategy
- Efficient sampling for non-orthogonal bases

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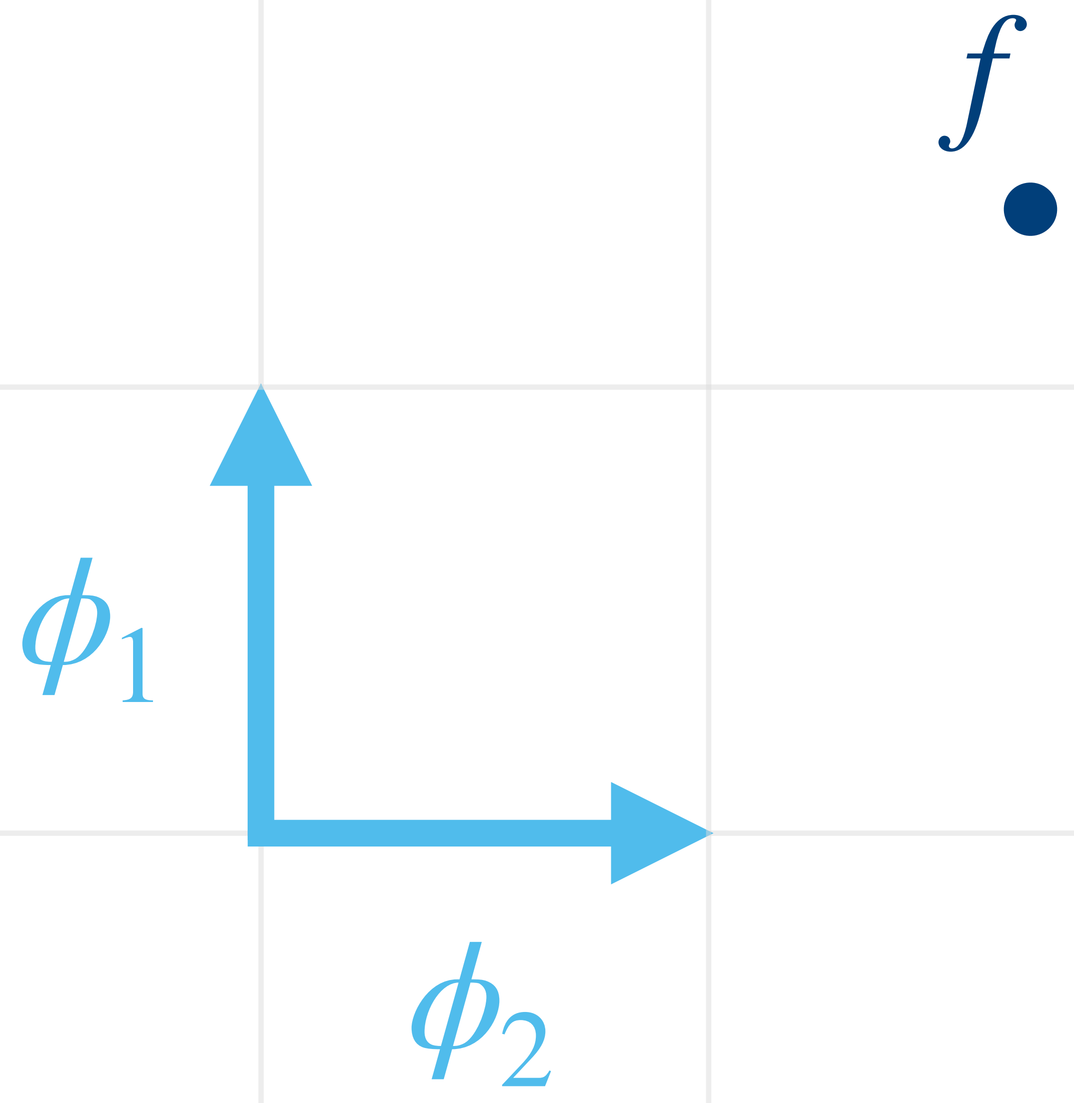
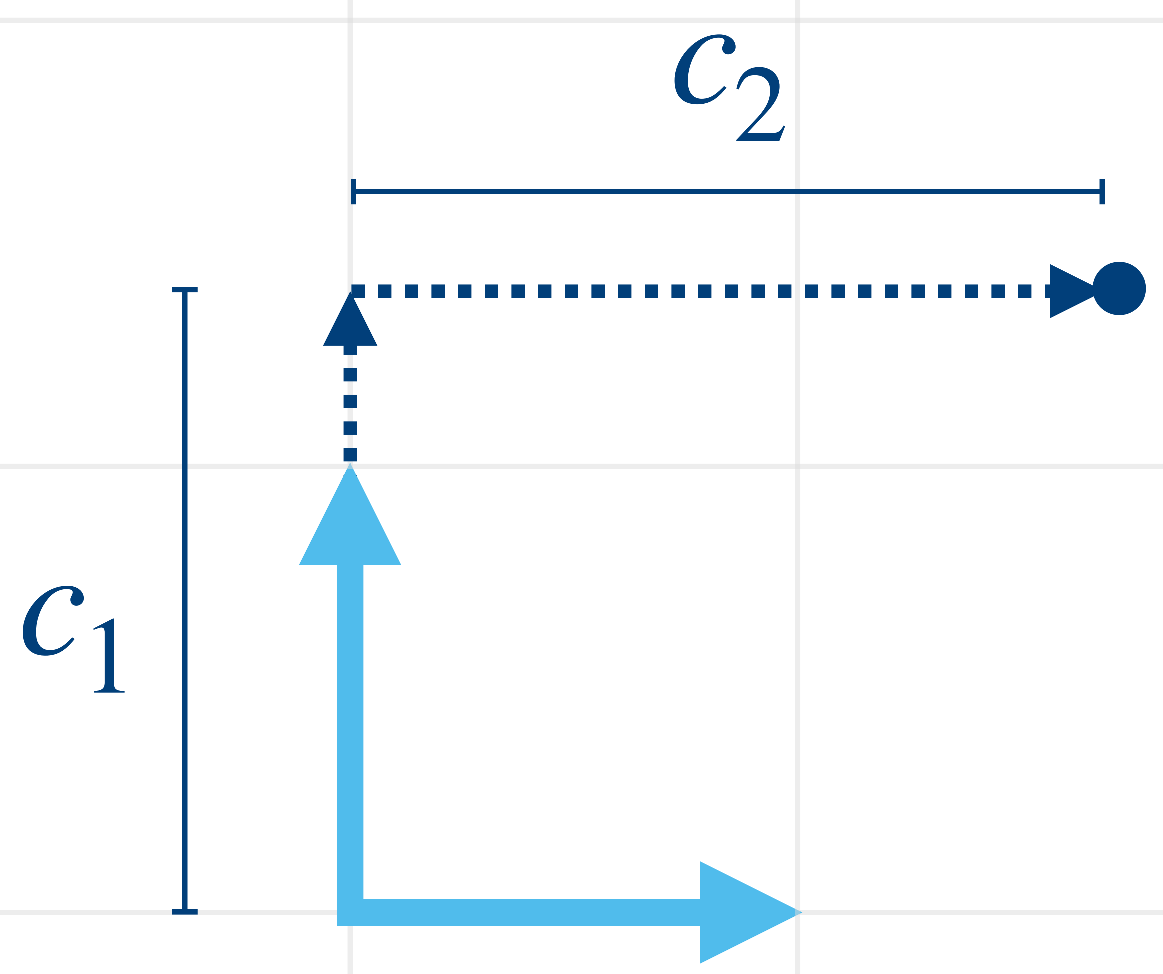
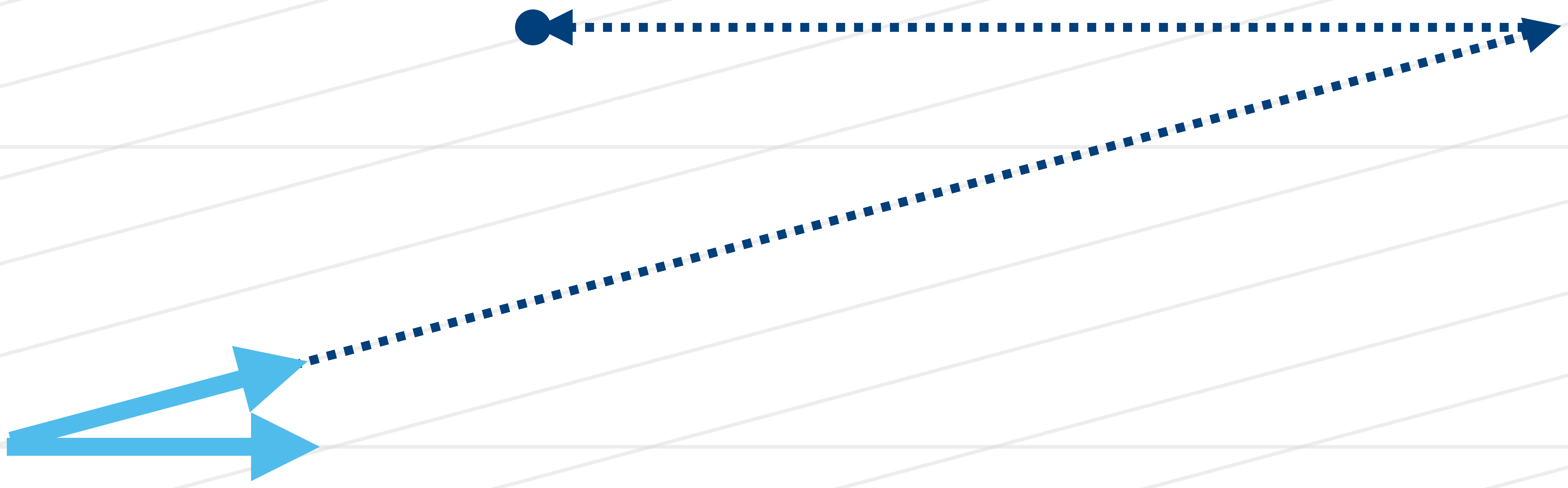
Approximation problem:

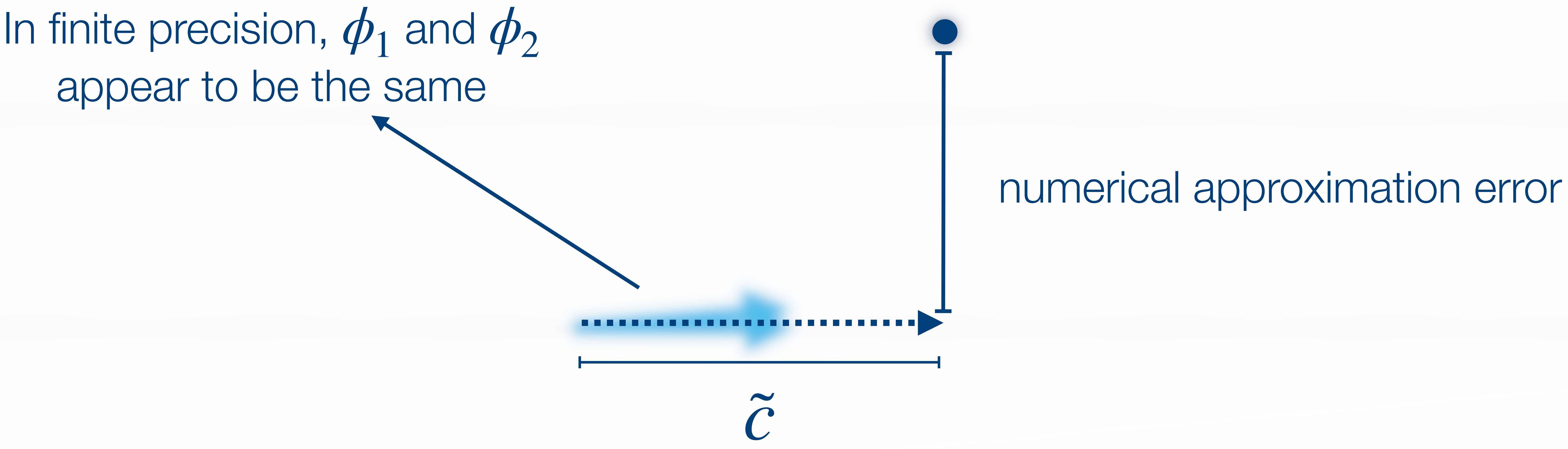
given f find c such that $\left\| f - \sum_j c_j \phi_j \right\|_{L^2(X)}$ is small




c_2
c_1



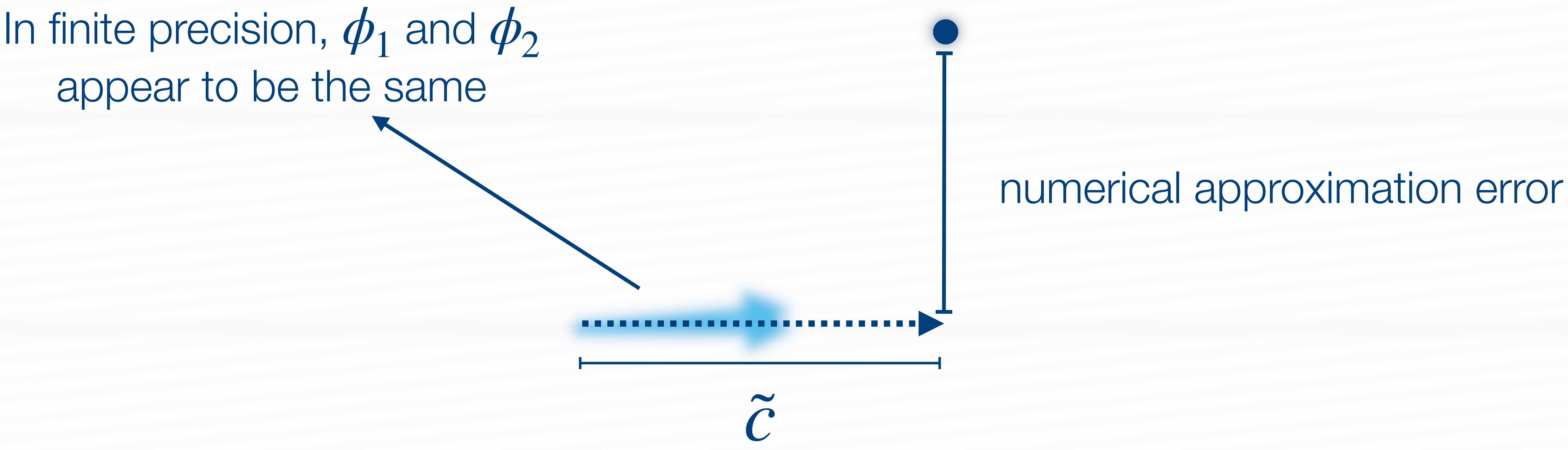
In finite precision, ϕ_1 and ϕ_2 appear to be the same


In finite precision, ϕ_1 and ϕ_2 appear to be the same
numerical approximation error
$\tilde{c}$

Rounding errors can result in

- a loss of accuracy
- a decrease in required data

compared to the "analytical case"



Rounding errors can result in

- a loss of accuracy (Adcock, Huybrechs 2019/2020)
- a decrease in required data (H., Huybrechs 2025)

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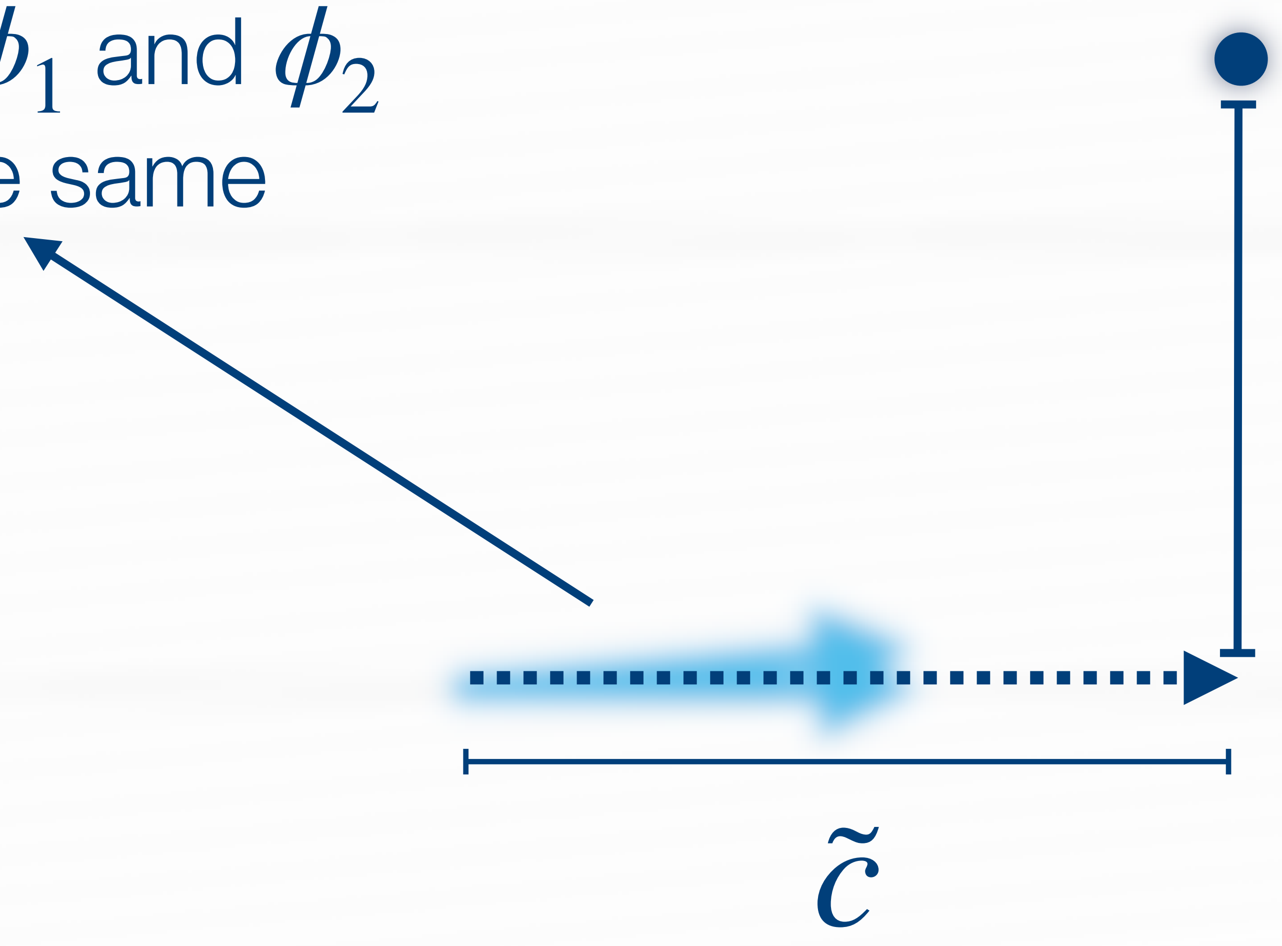

Numerical approximation

$$c_d = \arg\min_{c \in \mathbb{C}^n} \| \mathscr{M}(\mathscr{T} c - f) \|_2^2$$

where

- $\mathscr{T} \ : \ c \mapsto \sum_{i=1}^{n} c_i \phi_i$ is the synthesis operator
- $\mathscr{M} \ : \ v \mapsto \begin{bmatrix} \sqrt{w_1} v(x_1) & \dots & \sqrt{w_m} v(x_m) \end{bmatrix}^{\mathsf{T}}$ is a sampling operator

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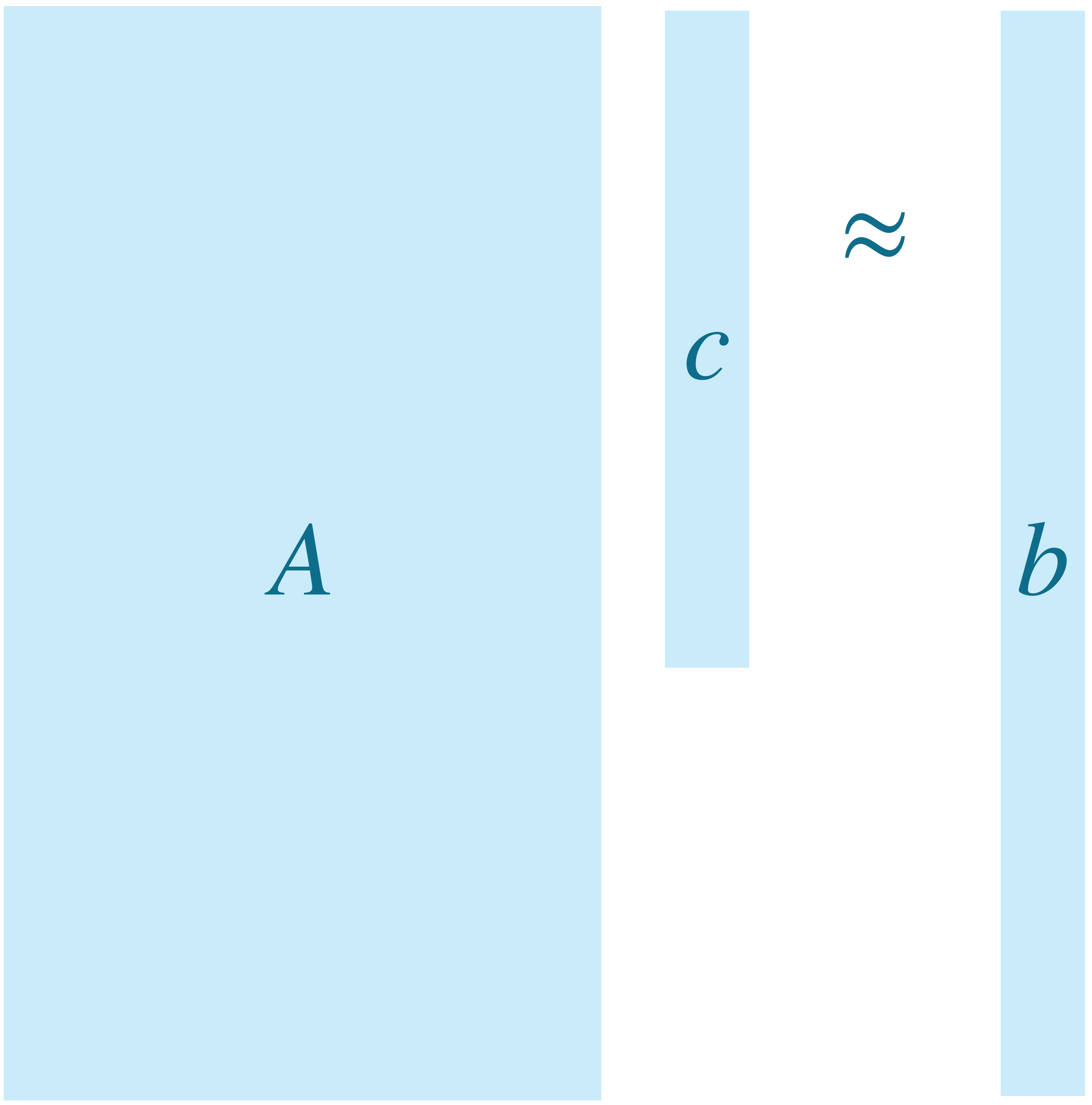

$A_{ij} = \sqrt{w_i} \phi_j(x_i)$ $\qquad$ $\sqrt{w_i} f(x_i)$

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When $\|\mathscr{M} \cdot \|_2 \approx \| \cdot \|_{L^2(X)}$, we get

$$\left\| \mathscr{T} c_d - f \right\|_{L^2(X)} \lesssim \min_{c \in \mathbb{C}^n} \left\| \mathscr{T} c - f \right\|_{L^2(X)}$$

Numerical approximation

$$\tilde{c}_d = \arg\min_{c \in \mathbb{C}^n} \underbrace{\left\| \mathscr{M}(\mathscr{T} c - f) \right\|_2^2}_{\| Ac - b \|_2^2} + \epsilon^2 \|c\|_2^2$$

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- $\epsilon \propto \epsilon_{\text{mach}}$ is due to finite-precision arithmetic

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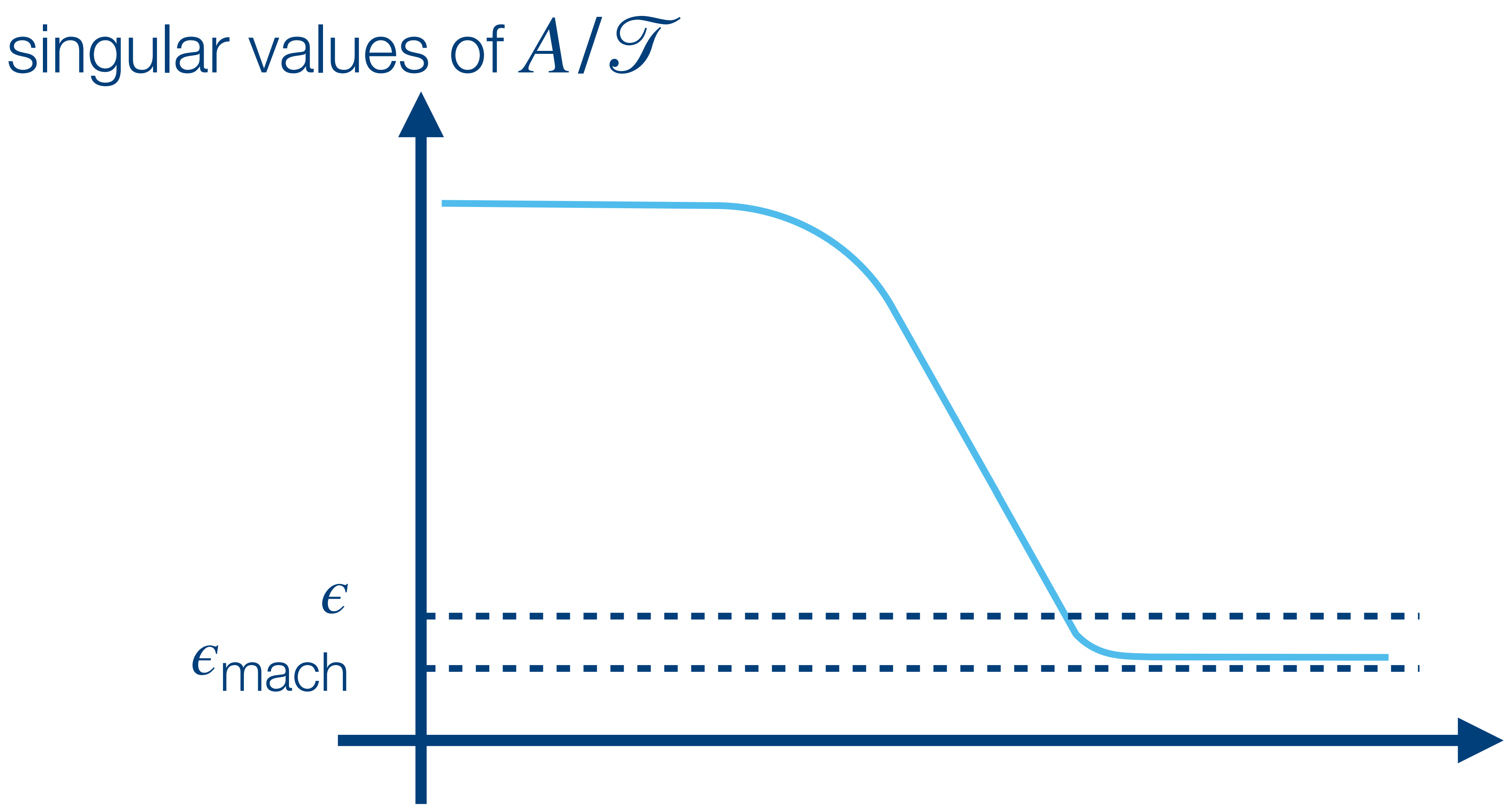

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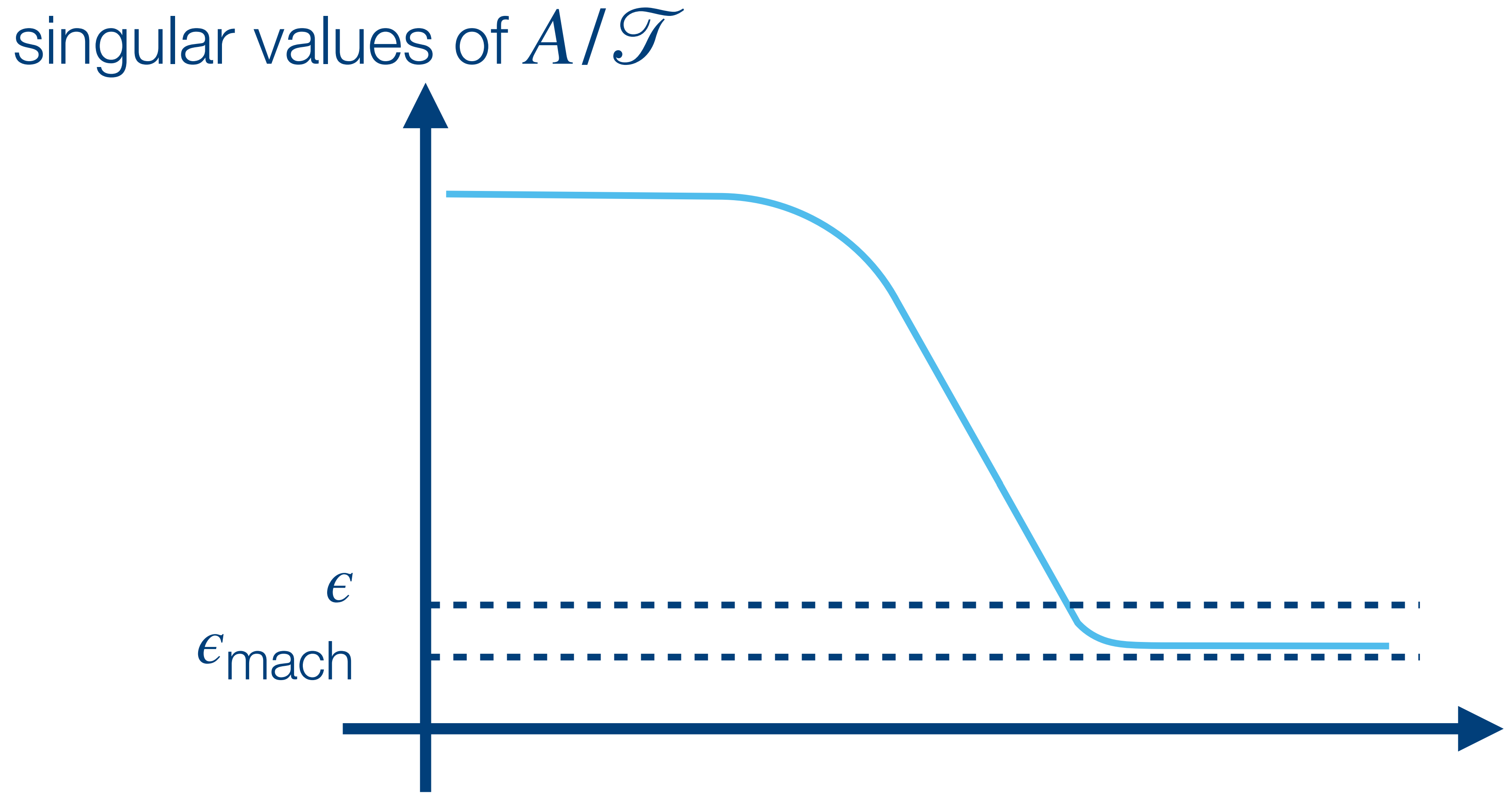

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(Adcock, Huybrechs 2019/2020)

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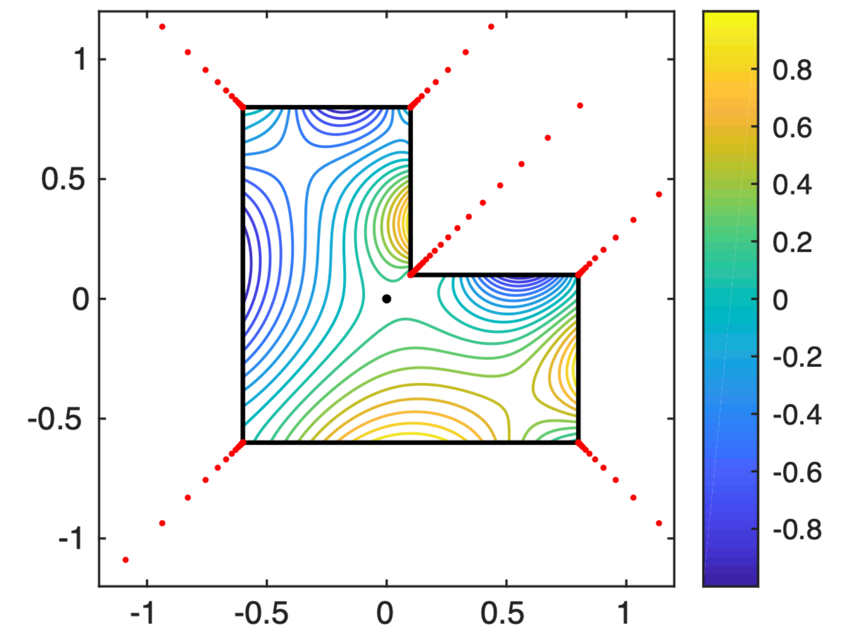

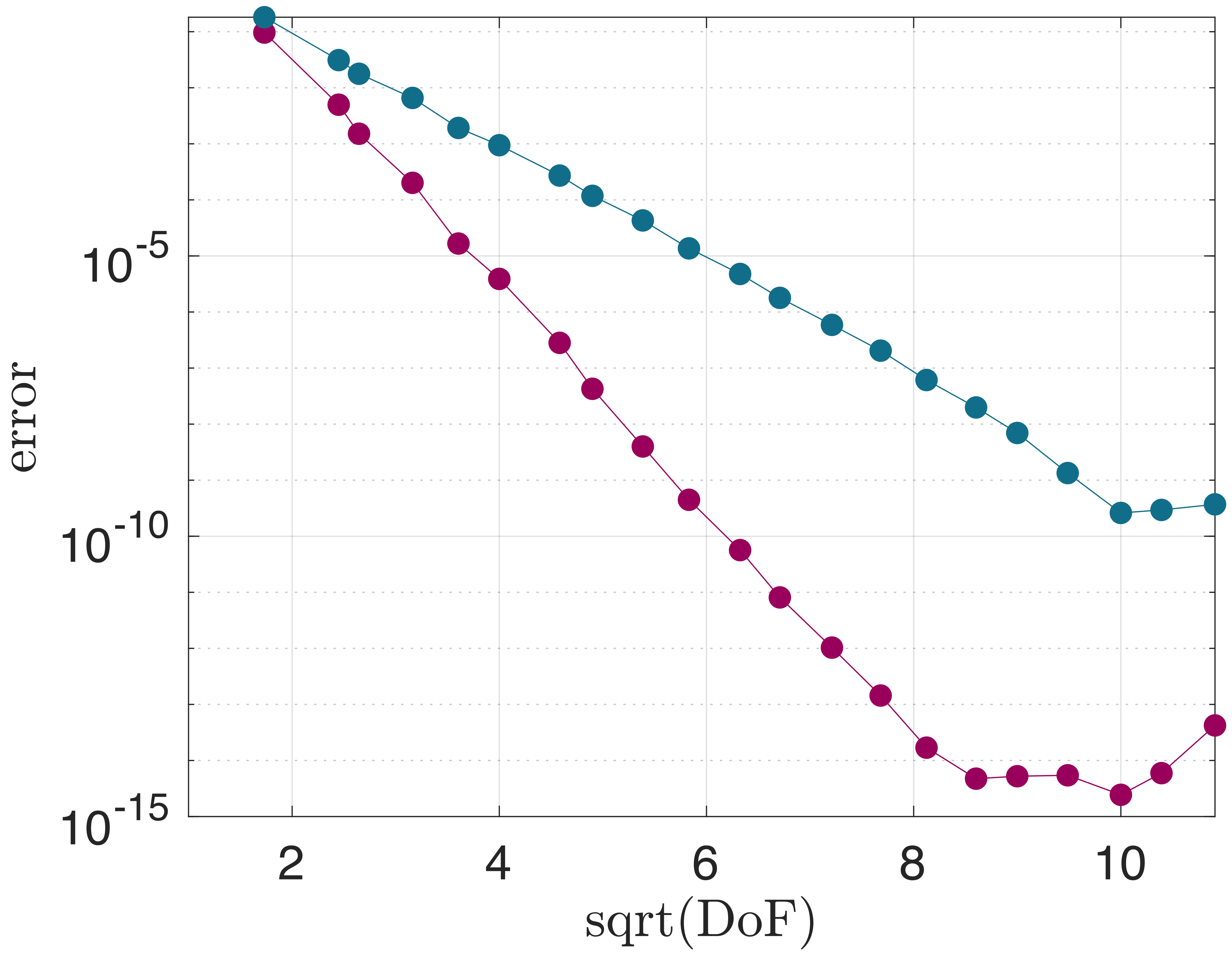

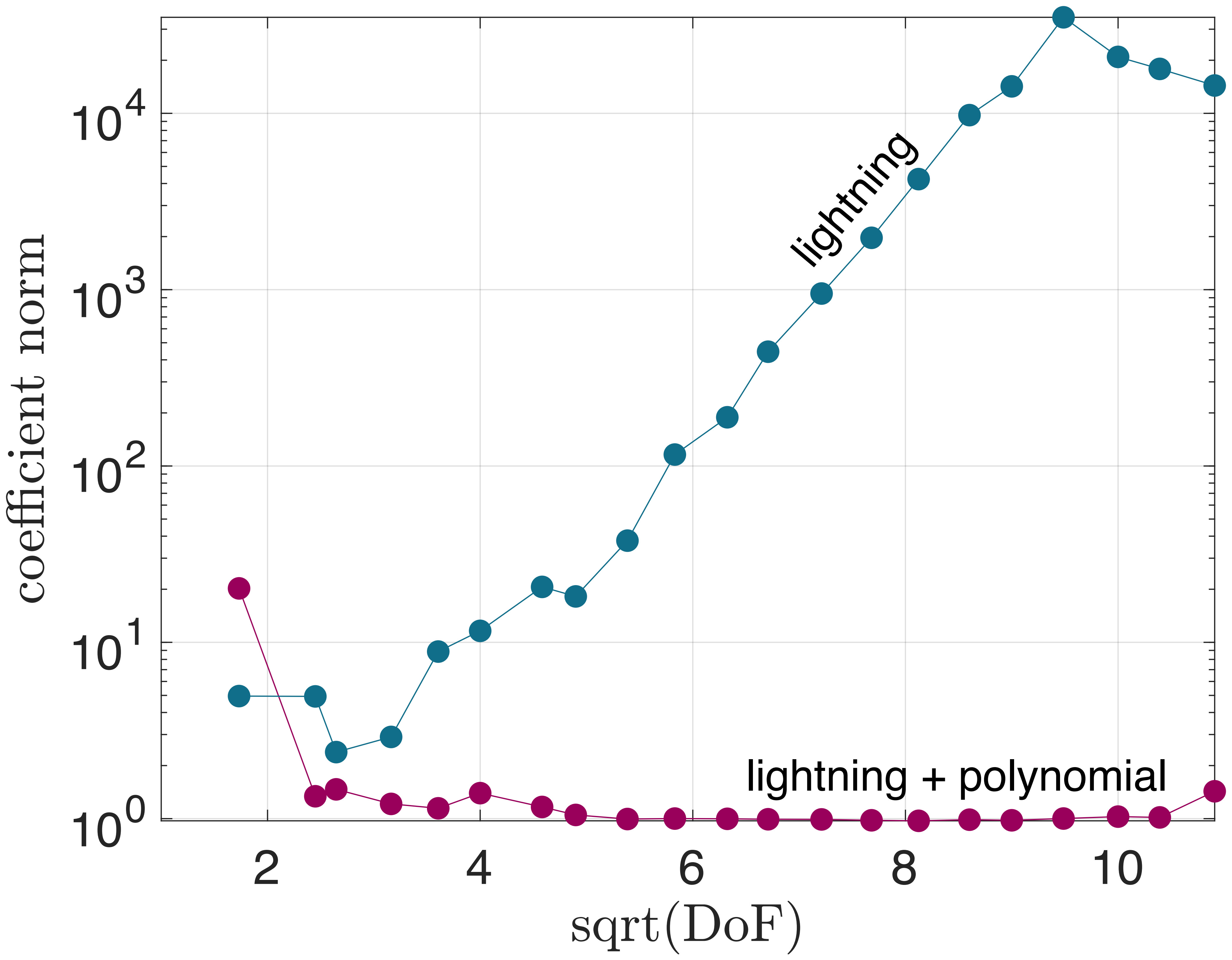

(H., Huybrechs and Trefethen, 2023)

- Approximation theory in finite precision
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We want

$$\|\mathscr{M}v\|_2^2 = \sum_{i=1}^{m} w_i |v(x_i)|^2 \qquad \approx \qquad \|v\|_{L^2(X)}^2 = \int_X v^2 dx, \qquad \forall v \in V = \text{span}(\{\phi_i\}_{i=1}^n)$$

using as few sample points $m \geq n$ as possible.

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I.e., which points are more important than others?

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→ Every $v \in V$ should be visible on the grid, also functions that spike locally.

How much can a function spike around x?

We want

$$\|\mathscr{M}v\|_2^2 = \sum_{i=1}^{m} w_i |v(x_i)|^2 \qquad \approx \qquad \|v\|^2_{L^2(X)} = \int_X v^2 dx, \qquad \forall v \in V = \mathrm{span}(\{\phi_i\}_{i=1}^n)$$

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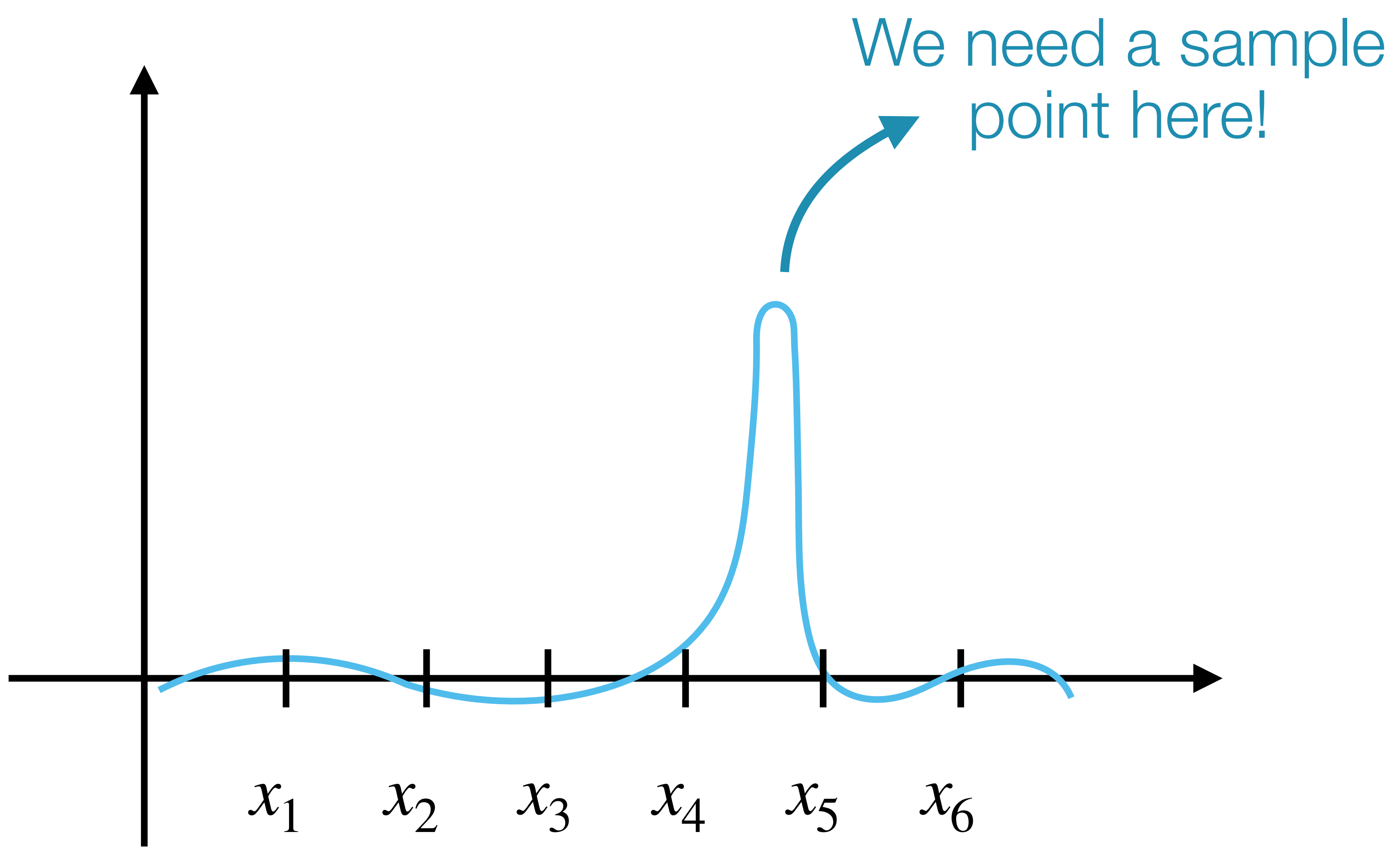

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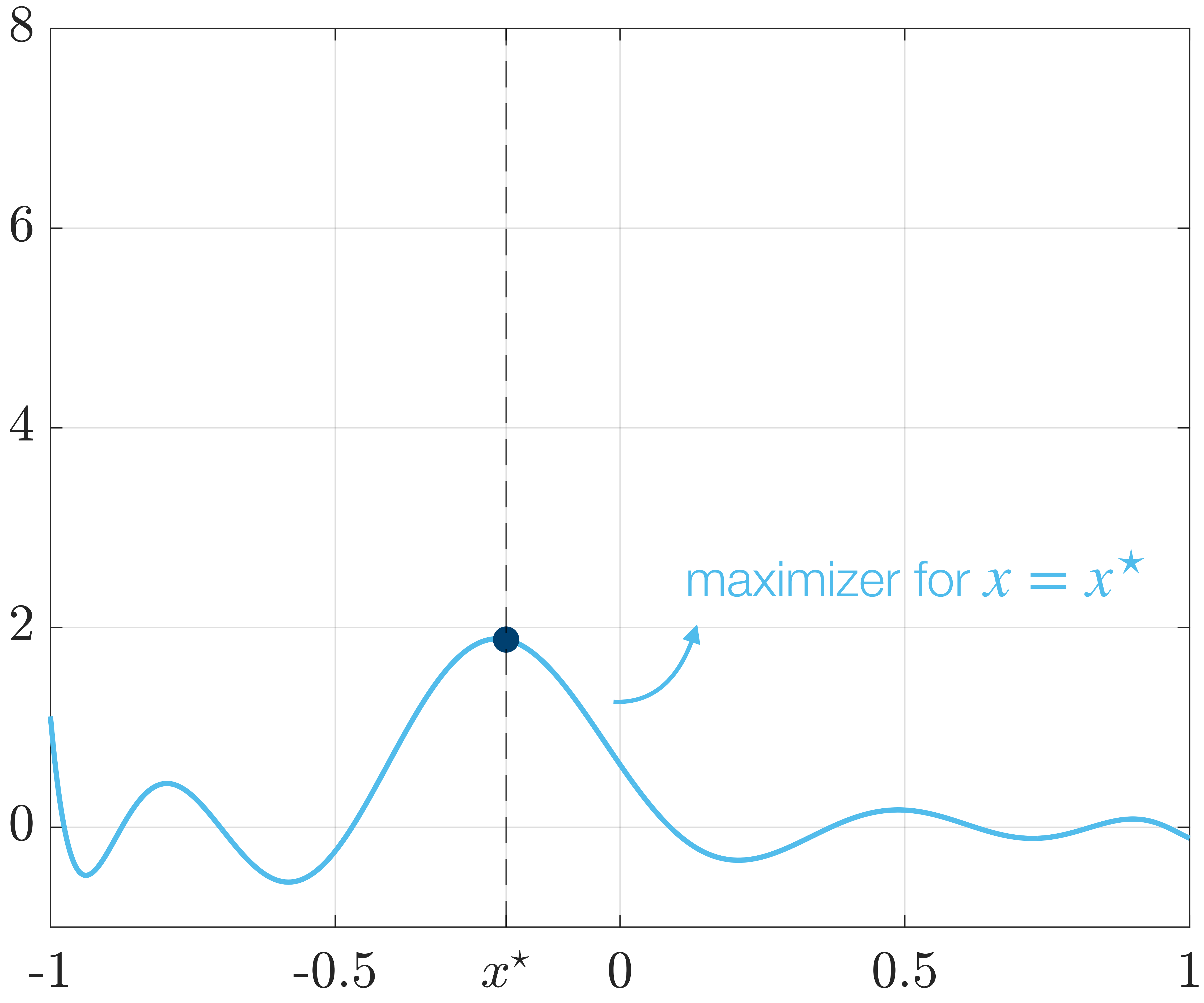
<u>How much can a function spike around x?</u>

We can quantify how much functions can spike around x by

$$\max_{v \in V, \|v\|_{L^2(X)}=1} |v(x)|$$

→ let’s look at polynomials up to degree 10



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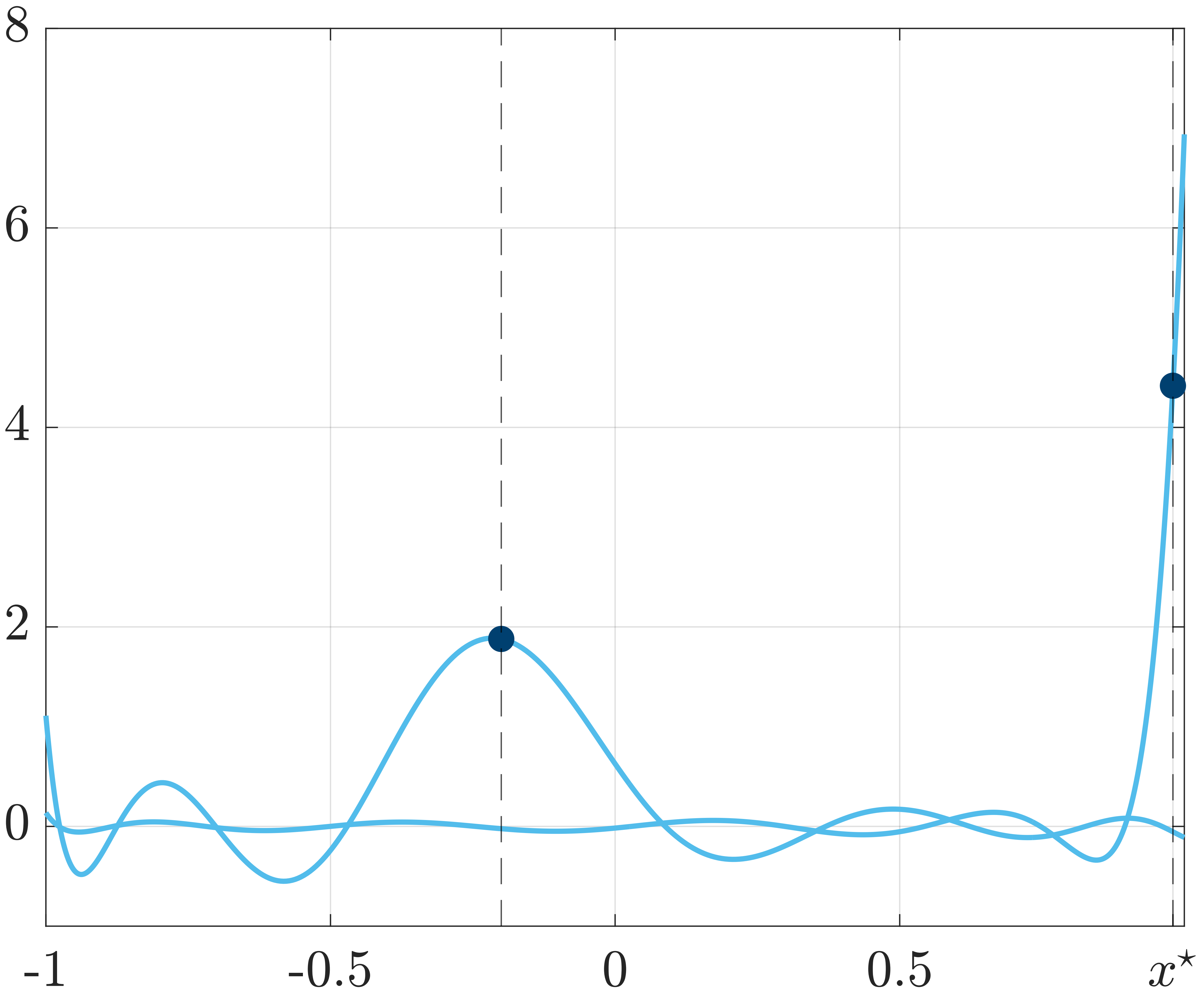

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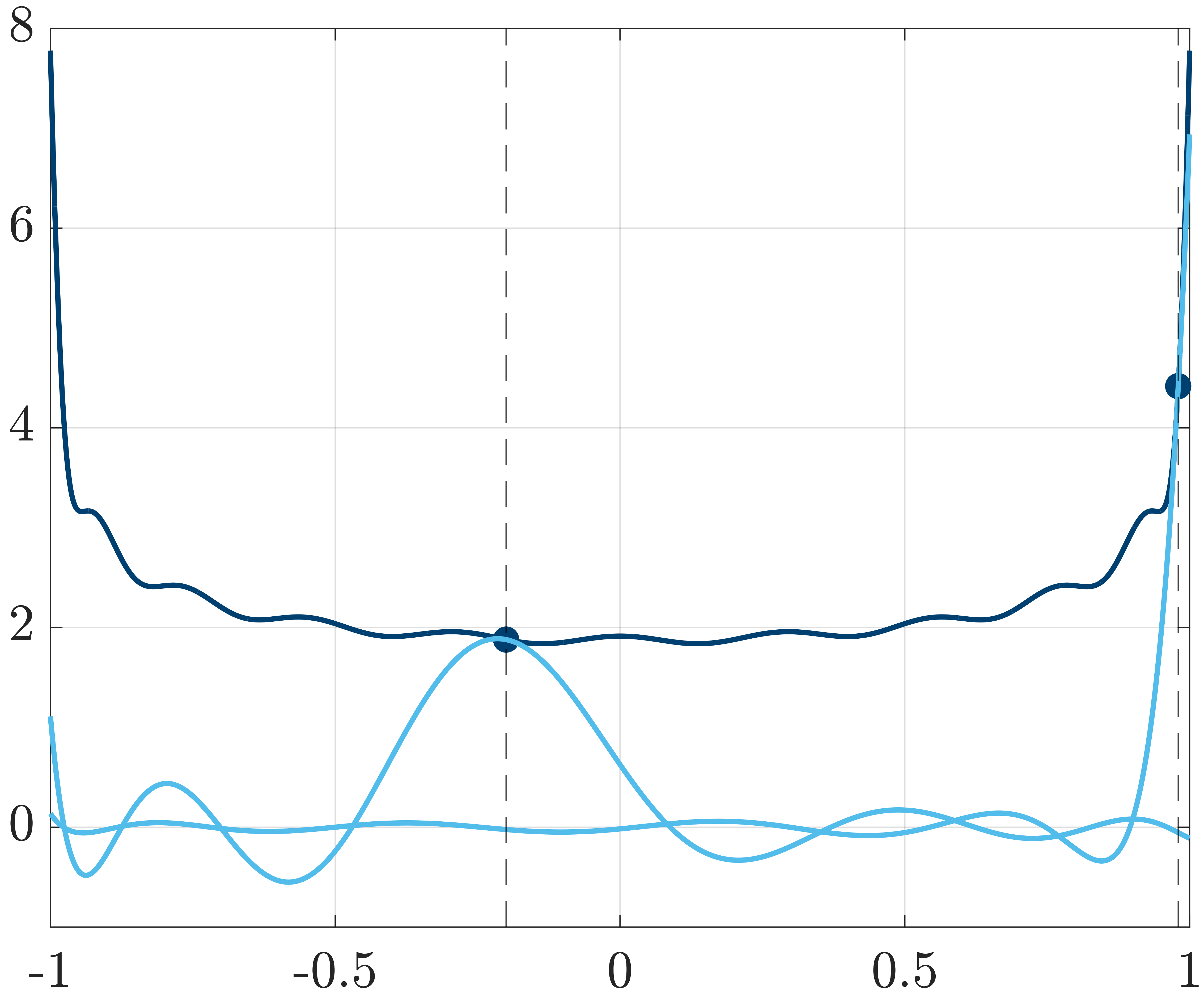

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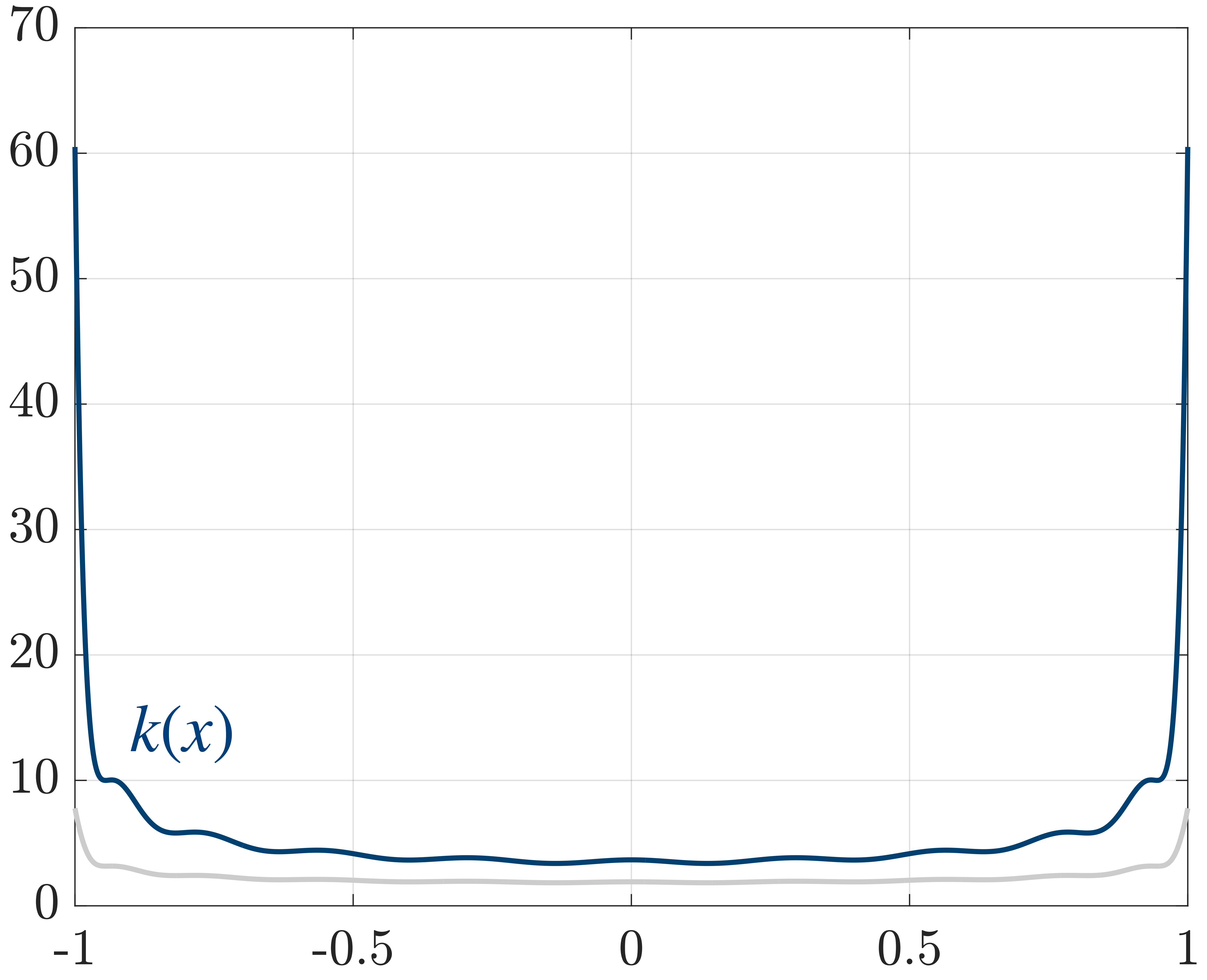

We can quantify how much functions can spike around x by

$$k(x) = \max_{v \in V, \|v\|_{L^2(X)}=1} |v(x)|^2$$

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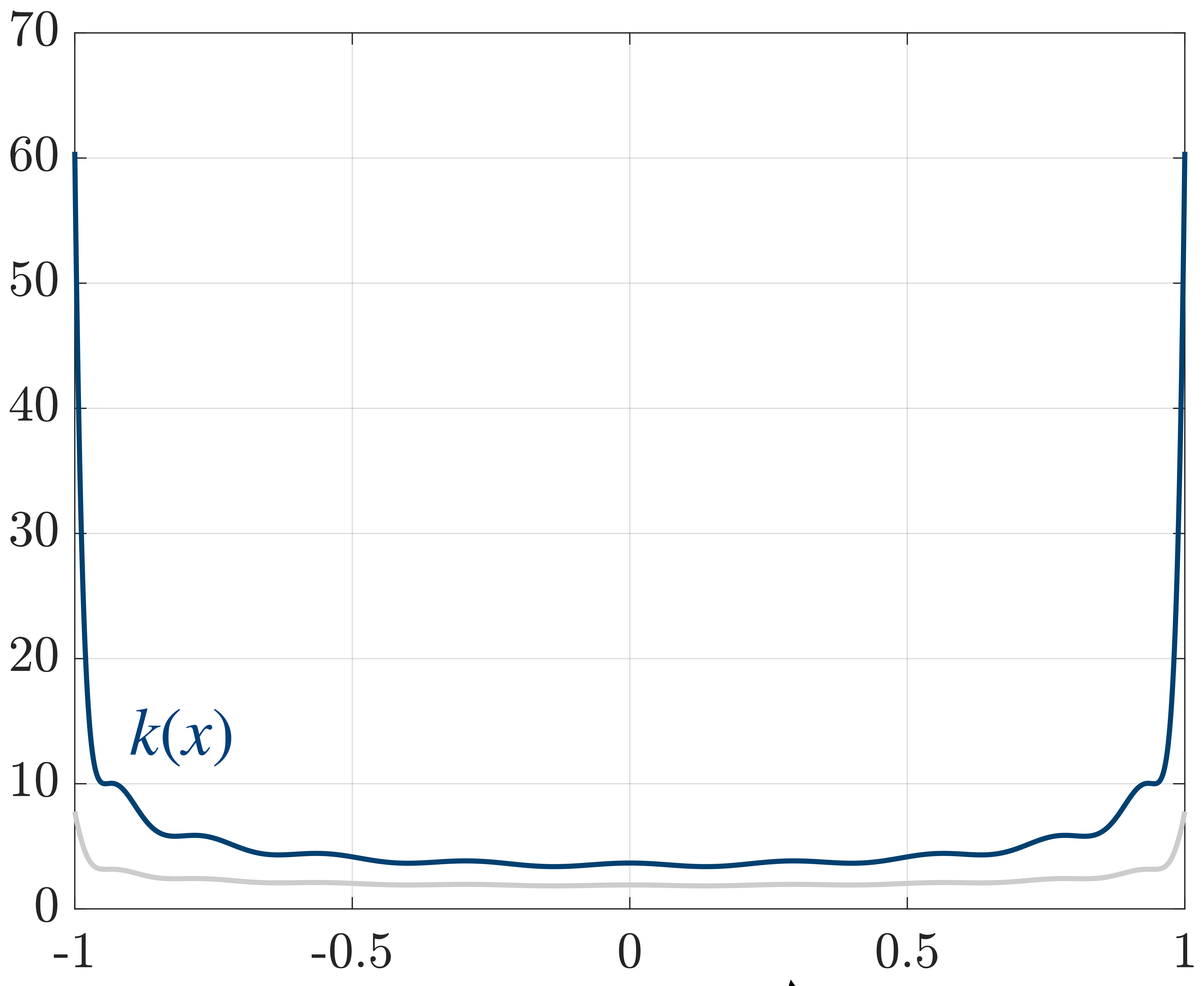

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With increasing polynomial degree, $k(x)$ converges to the arcsine measure (after normalization)

Christoffel function

The function

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is known as the inverse of the Christoffel function

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- $k(x) = \sum_{i=1}^{n} |u_i(x)|^2$ where $\{u_i\}_{i=1}^{n}$ is an orthonormal basis for V

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$$(G)_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2(X)}$$

Christoffel sampling

(Cohen and Migliorati, 2017)

If one draws $m = \mathcal{O}(n \log(n))$ samples according to

$$d\mu = w\,dx \qquad \text{with} \;\; w(x) = k(x)/n$$

then, with high probability,

$$\| \mathcal{T} c_d - f \|_{L^2(X)} \lesssim \min_{c \in \mathbb{C}^n} \| \mathcal{T} c - f \|_{L^\infty(X)}$$

for the weighted discrete least squares approximation

$$c_d = \arg\min_{c \in \mathbb{C}^n} \| \mathcal{M}(\mathcal{T} c - f) \|_2^2$$

Christoffel sampling

corresponds to

$$\mathscr{M} : v \mapsto \left[\sqrt{w_1}v(x_1) \quad \dots \quad \sqrt{w_m}v(x_m)\right]^{\mathsf{T}}$$

with $x_i \sim \mu$ and $w_i = w(x_i)/m$

(Cohen and Migliorati, 2017)

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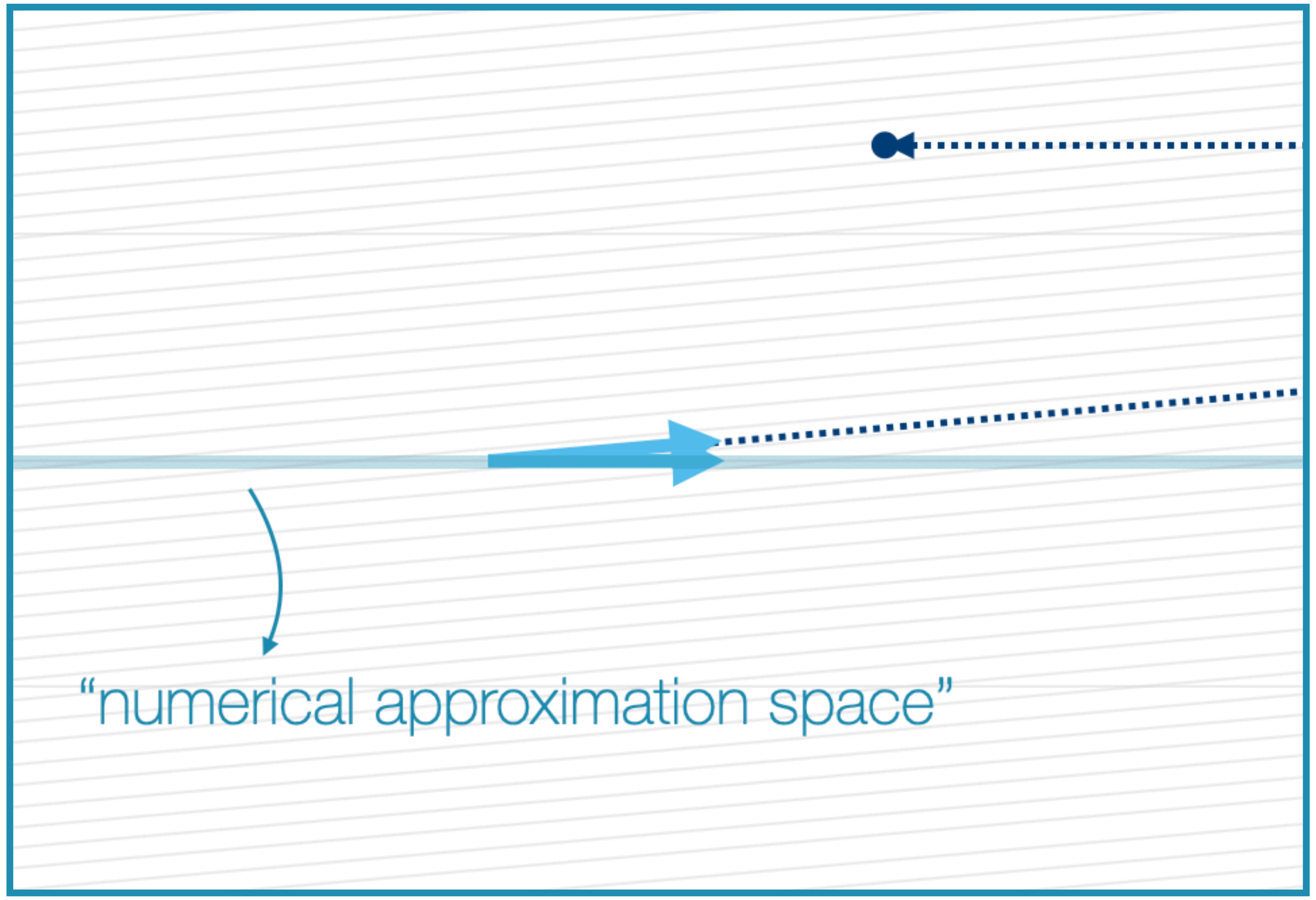
Influence of finite precision



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where $\epsilon \propto \epsilon_{\text{mach}}$ is due to finite-precision arithmetic

(H. and Huybrechs, 2025)

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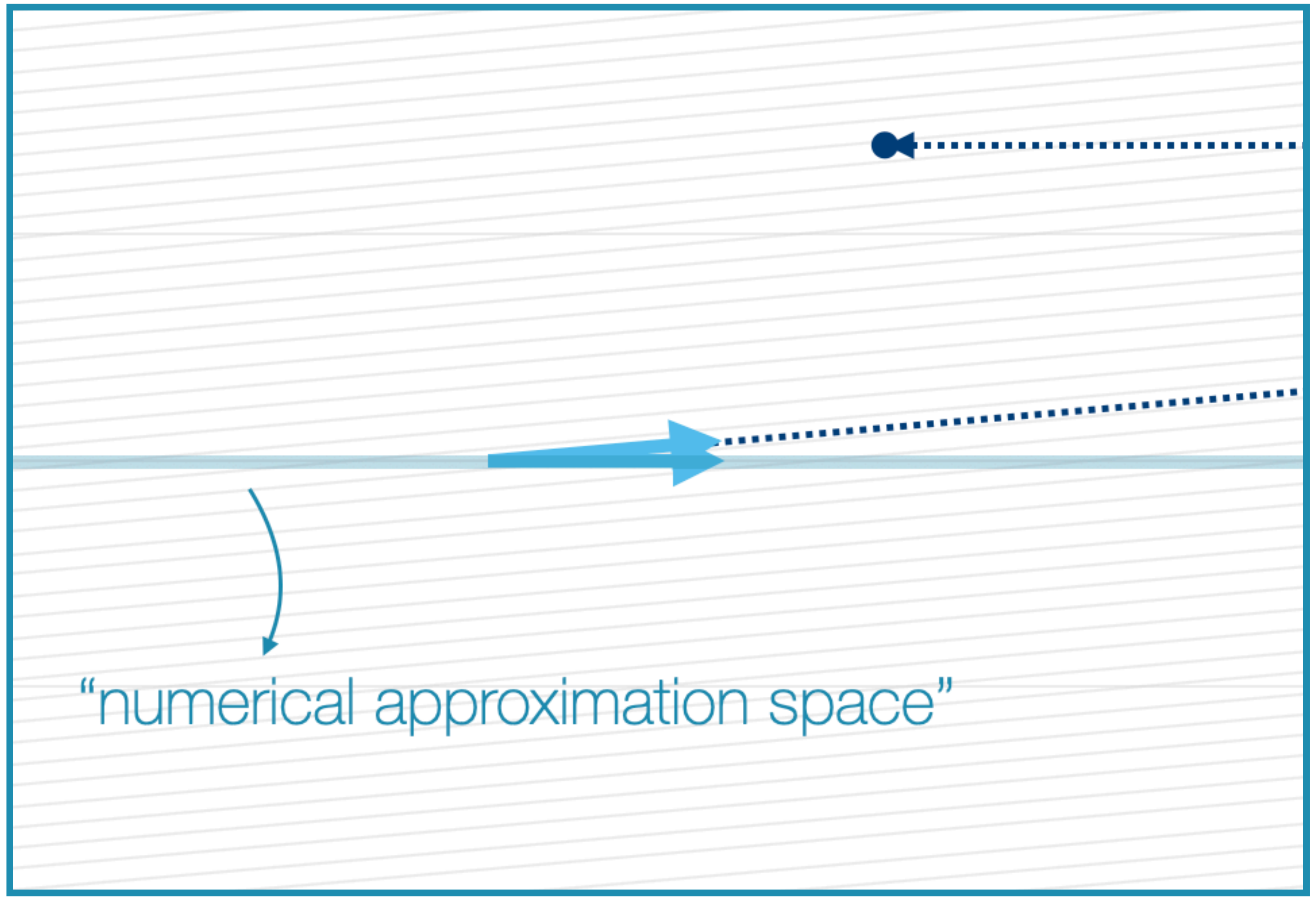

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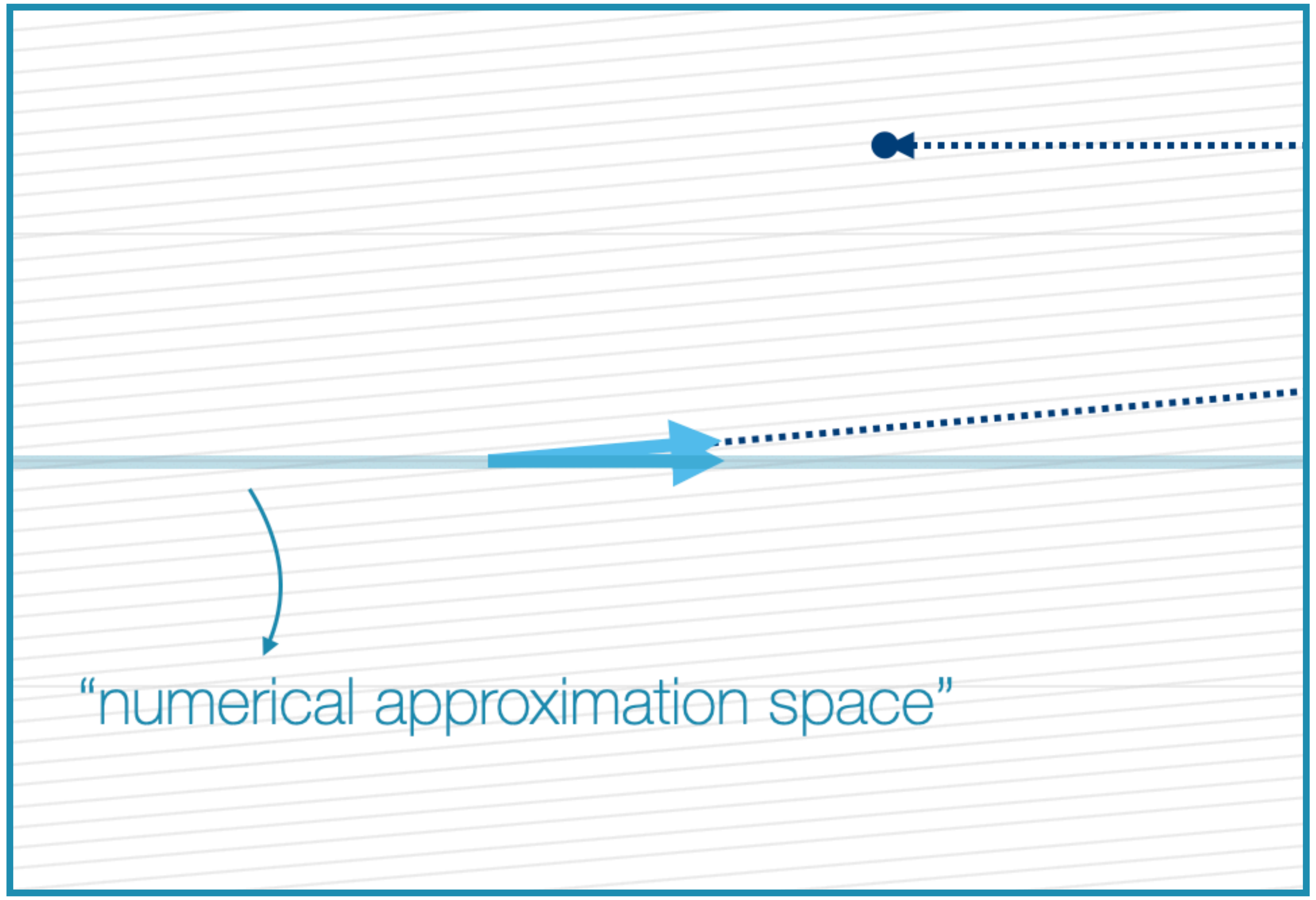

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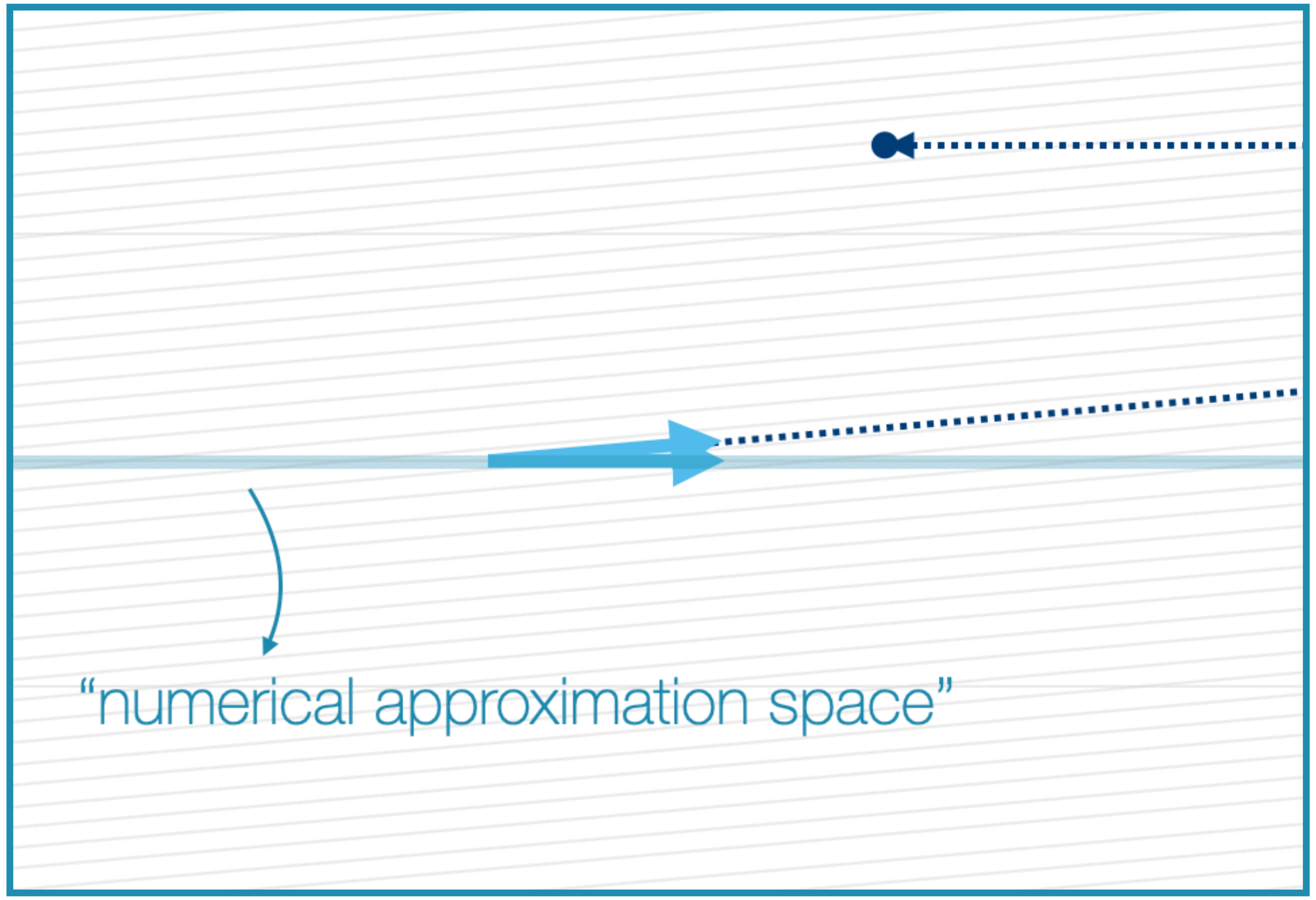
where $\epsilon \propto \epsilon_{\text{mach}}$ is due to finite-precision arithmetic

$$k(x) = \max_{v \in V, v \neq 0} \frac{|v(x)|^2}{\|v\|_{L^2}^2}$$

(H. and Huybrechs, 2025)

Influence of finite precision



$$\left\| \mathcal{T}\tilde{c}_d - f \right\|_{L^2(X)} \lesssim \min_{c\in\mathbb{C}^n} \left\| \mathcal{T}c - f \right\|_{L^2(X)} + \epsilon\|c\|_2$$

where $\epsilon \propto \epsilon_{\text{mach}}$ is due to finite-precision arithmetic

$$k(x) = \max_{c\in\mathbb{C}^n, \mathcal{T}c\neq 0} \frac{|\mathcal{T}c(x)|^2}{\|\mathcal{T}c\|_{L^2}^2}$$

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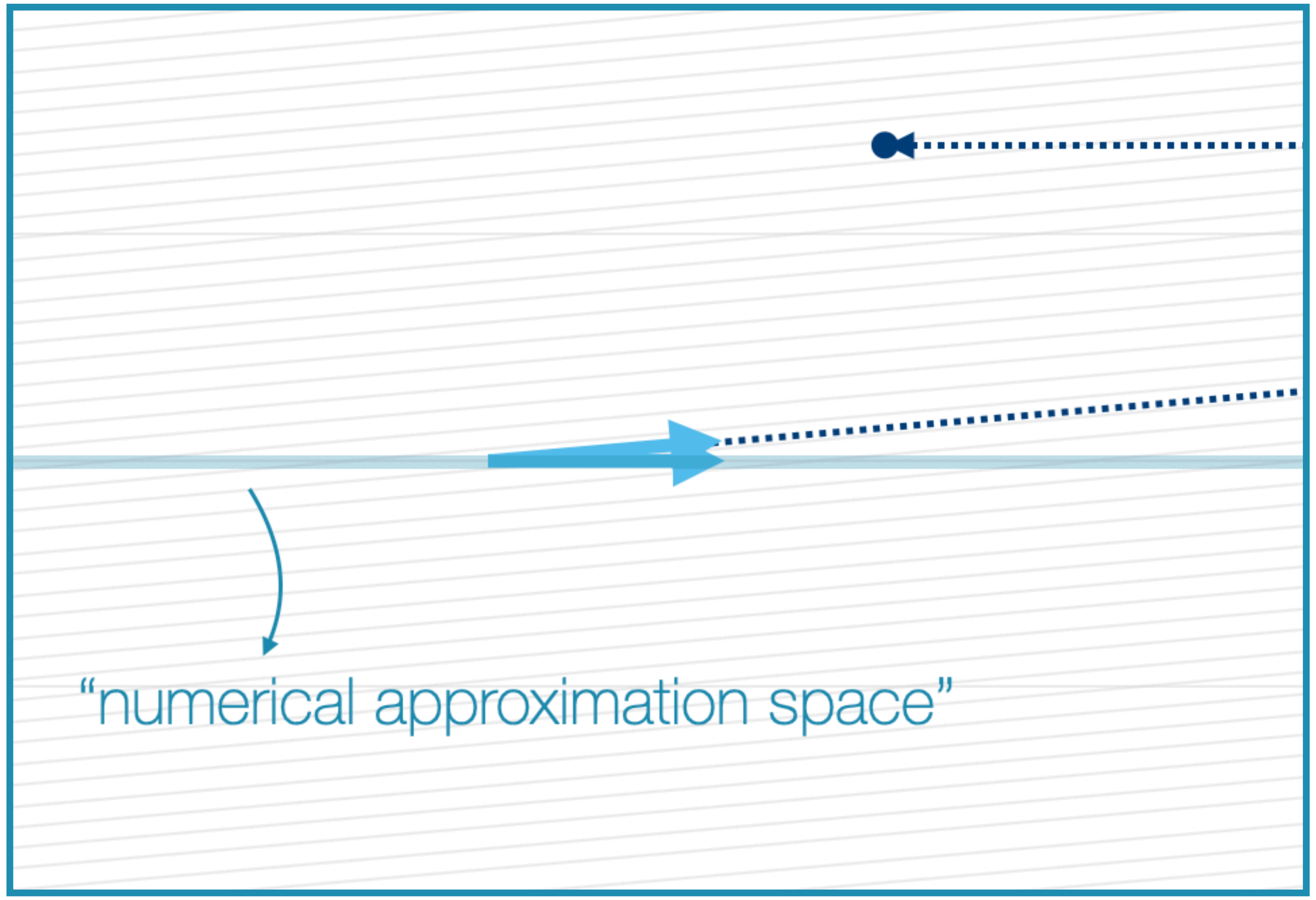

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$$k(x) = \max_{c \in \mathbb{C}^n, \mathcal{T}c \neq 0} \frac{|\mathcal{T}c(x)|^2}{\|\mathcal{T}c\|_{L^2}^2} \longrightarrow k^{\epsilon}(x) = \max_{c \in \mathbb{C}^n, \mathcal{T}c \neq 0} \frac{|\mathcal{T}c(x)|^2}{\|\mathcal{T}c\|_{L^2}^2 + \epsilon^2 \|c\|_2^2}$$

(H. and Huybrechs, 2025)

Numerical Christoffel function

$$k^{\epsilon}(x) = \max_{c \in \mathbb{C}^n, \mathscr{T}c \neq 0} \frac{|\mathscr{T}c(x)|^2}{\|\mathscr{T}c\|_{L^2}^2 + \boxed{\epsilon^2 \|c\|_2^2}} = \Phi(x)^* \left(G + \boxed{\epsilon^2 I}\right)^{-1} \Phi(x)$$

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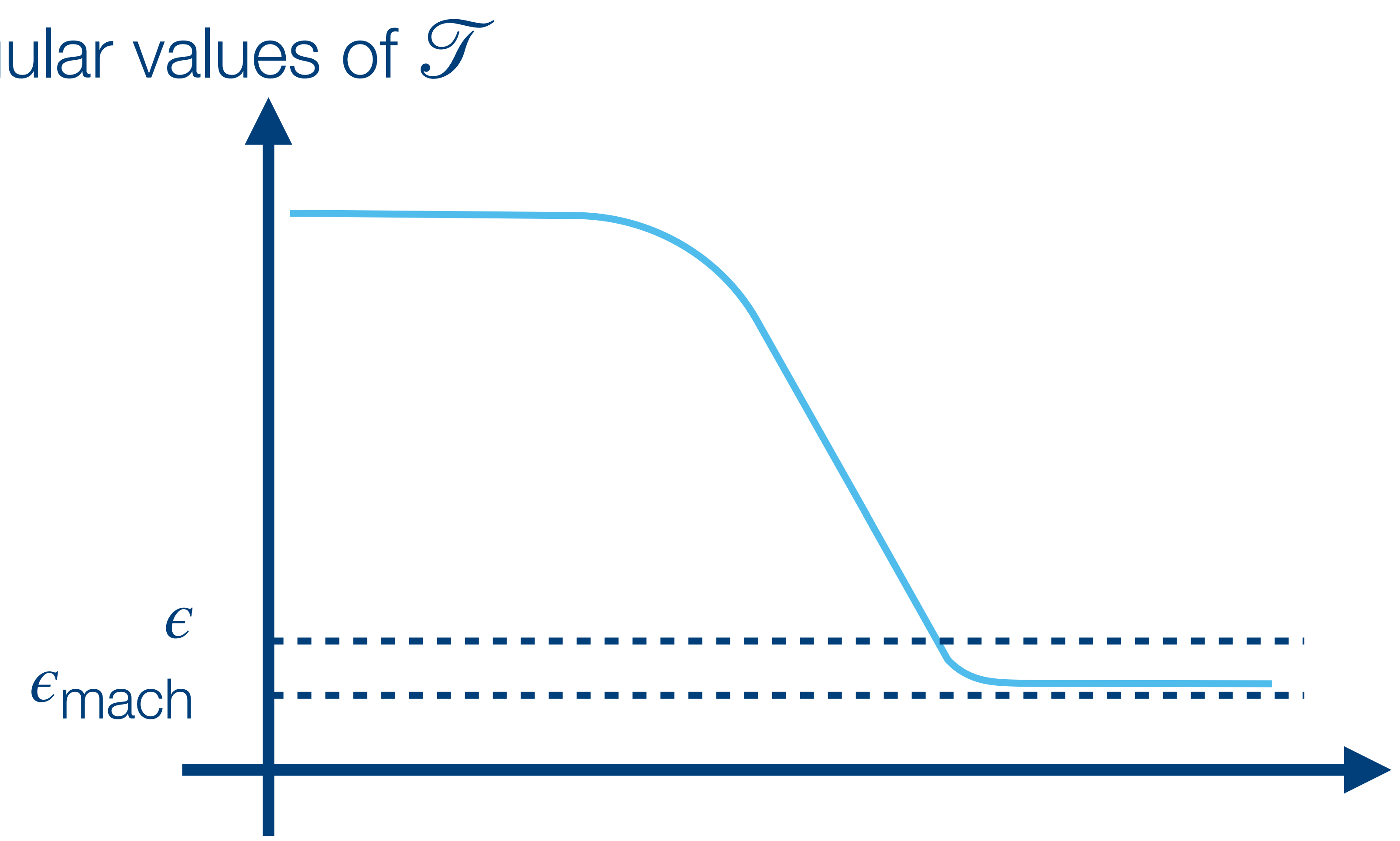
- $\int_X k^\epsilon dx = n^\epsilon$ where n^ϵ is the numerical dimension

$$n^\epsilon = \sum_{i=1}^{n} \frac{\sigma_i(\mathcal{T})^2}{\sigma_i(\mathcal{T})^2 + \epsilon^2}$$



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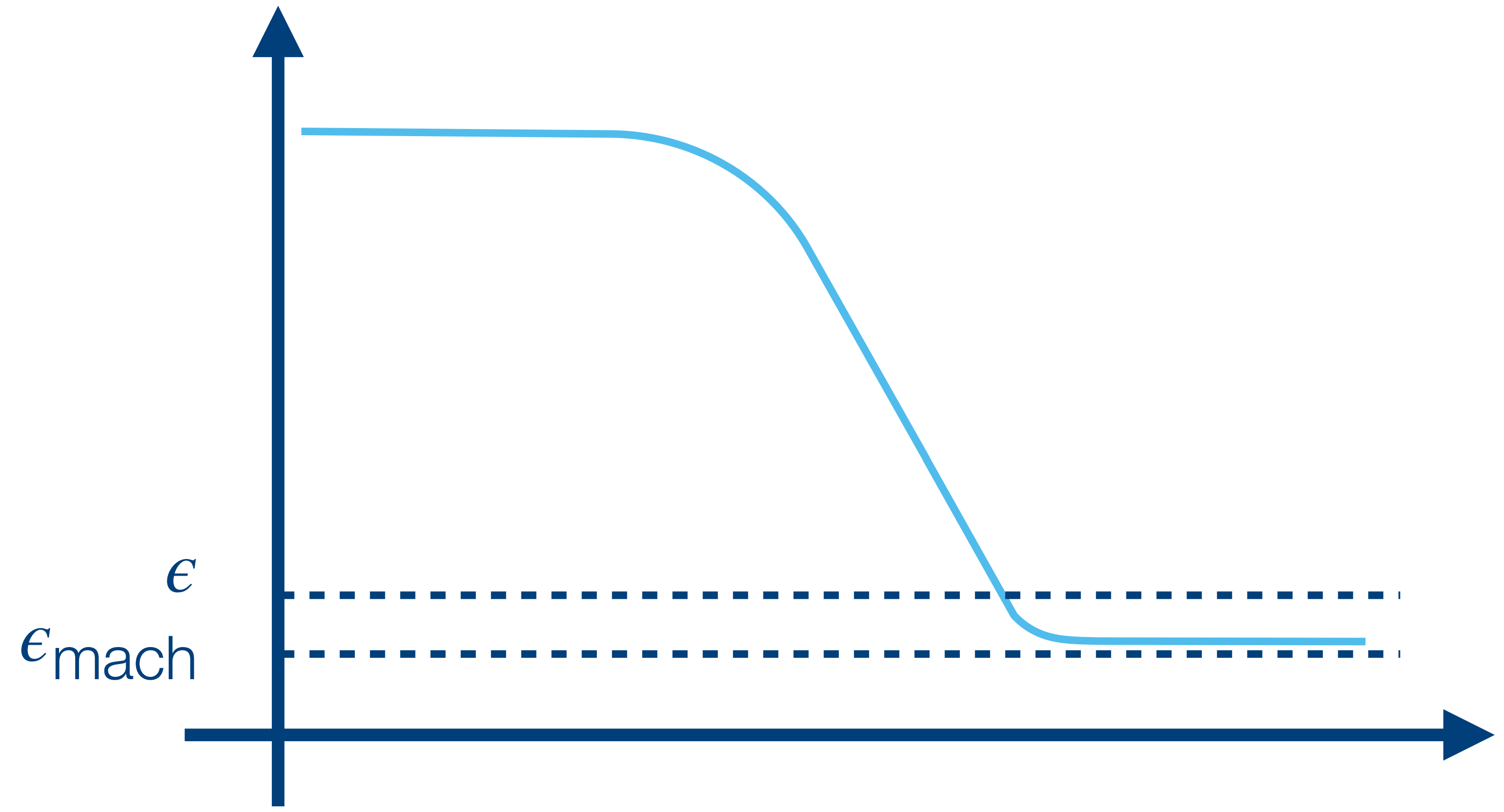

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$$n^\epsilon = \sum_{i=1}^{n} \frac{\sigma_i(\mathcal{T})^2}{\sigma_i(\mathcal{T})^2 + \epsilon^2}$$

$= n$ for an orthonormal basis

$\ll n$ for a heavily non-orthogonal basis

(H. and Huybrechs, 2025)

Numerical Christoffel sampling

(H. and Huybrechs, 2025)

If one draws $m = \mathcal{O}(n^\epsilon \log(n^\epsilon))$ samples according to

$$d\mu = w\,dx \qquad \text{with} \;\; w(x) = k^\epsilon(x)/n^\epsilon$$

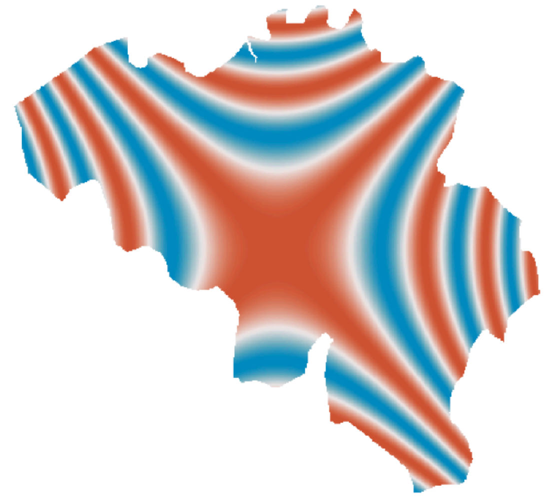
then, with high probability,

$$\left\| \mathcal{T}\tilde{c}_d - f \right\|_{L^2(X)} \lesssim \min_{c \in \mathbb{C}^n} \left\| \mathcal{T}c - f \right\|_{L^\infty(X)} + \epsilon \|c\|_2$$

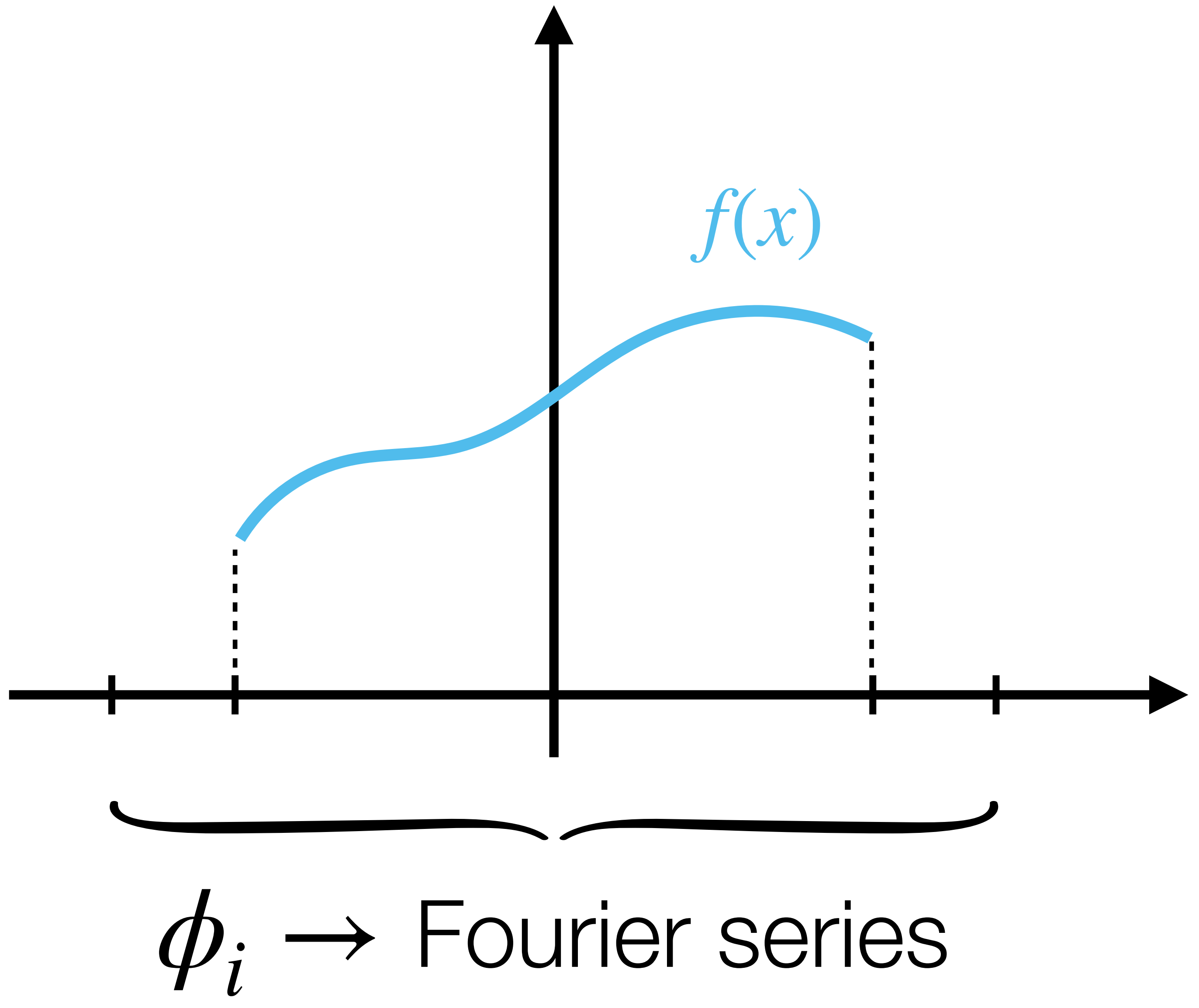
for the regularized weighted discrete least squares approximation

$$\tilde{c}_d = \arg\min_{c \in \mathbb{C}^n} \left\| \mathcal{M}(\mathcal{T}c - f) \right\|_2^2 + \epsilon^2 \|c\|_2^2$$

Fourier extension



= Fourier series restricted to a smaller domain
= non-orthogonal basis



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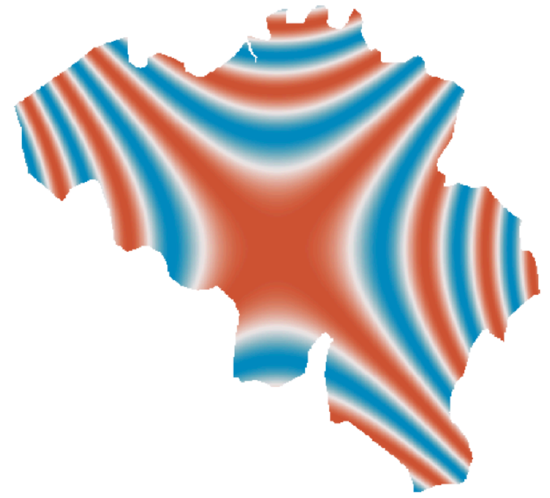
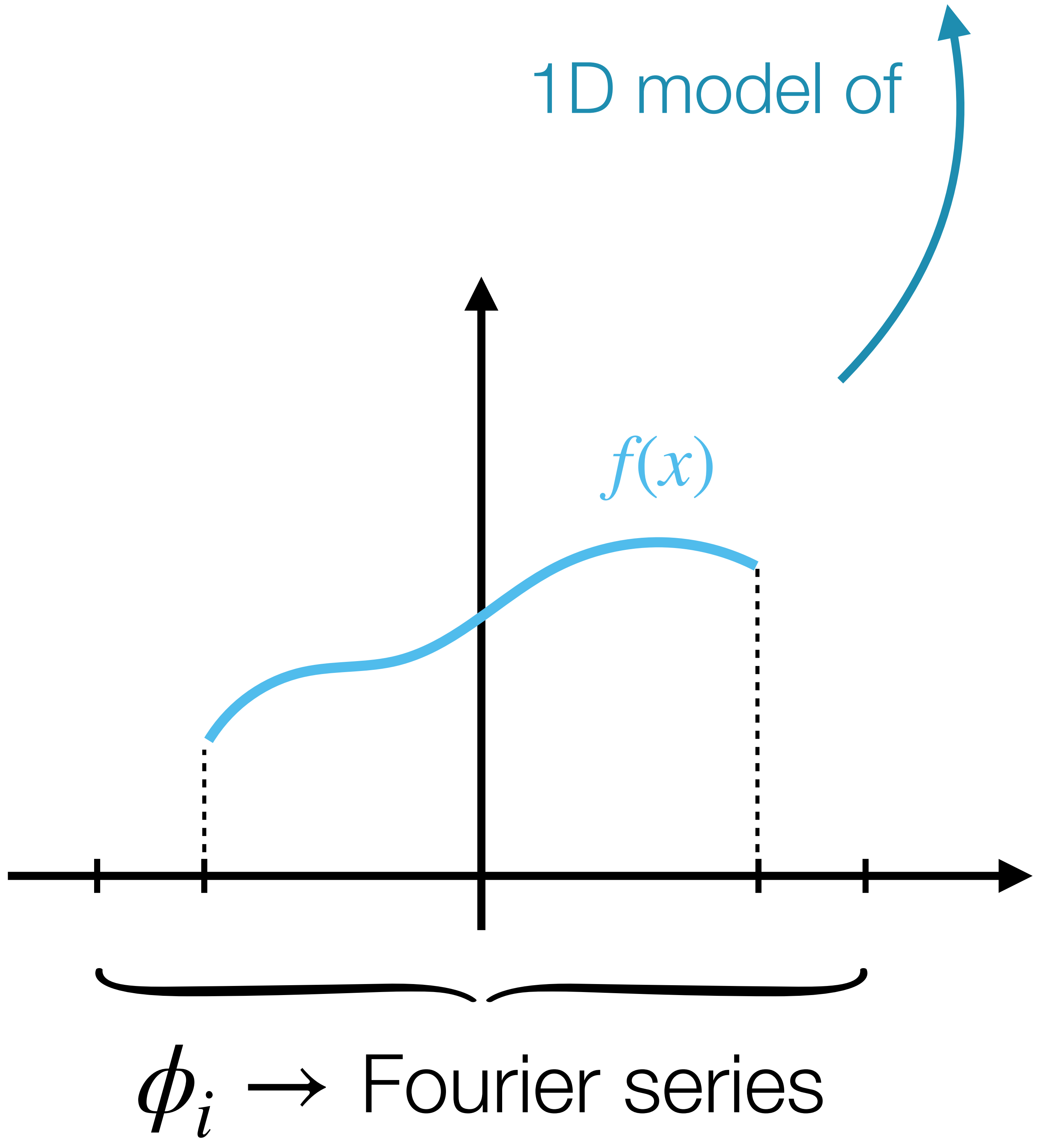


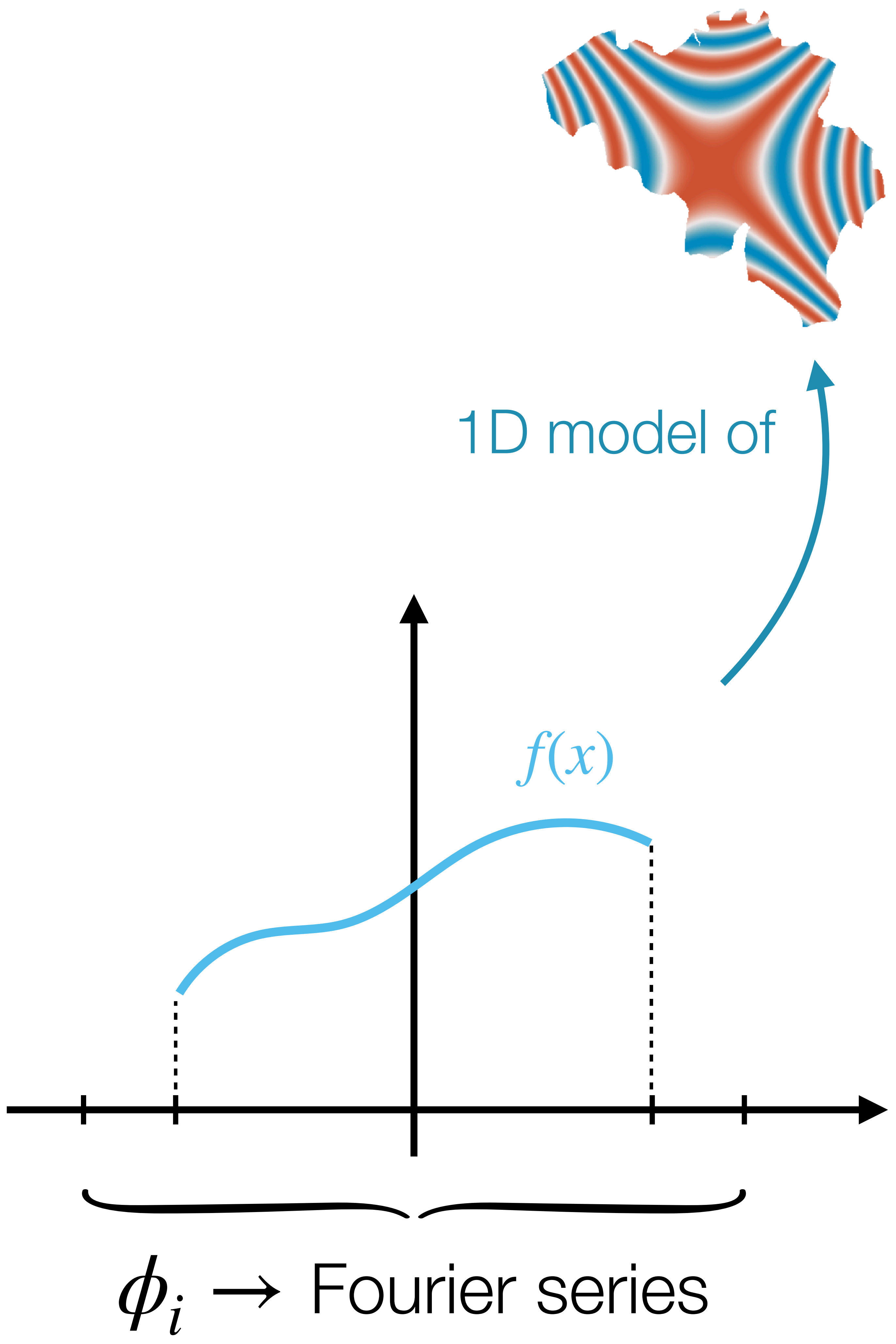


(H. and Huybrechs, 2025)

Fourier extension

= Fourier series restricted to a smaller domain
= non-orthogonal basis

$k(x)$ grows much larger near the boundaries than the middle → we need to cluster points there



(H. and Huybrechs, 2025)

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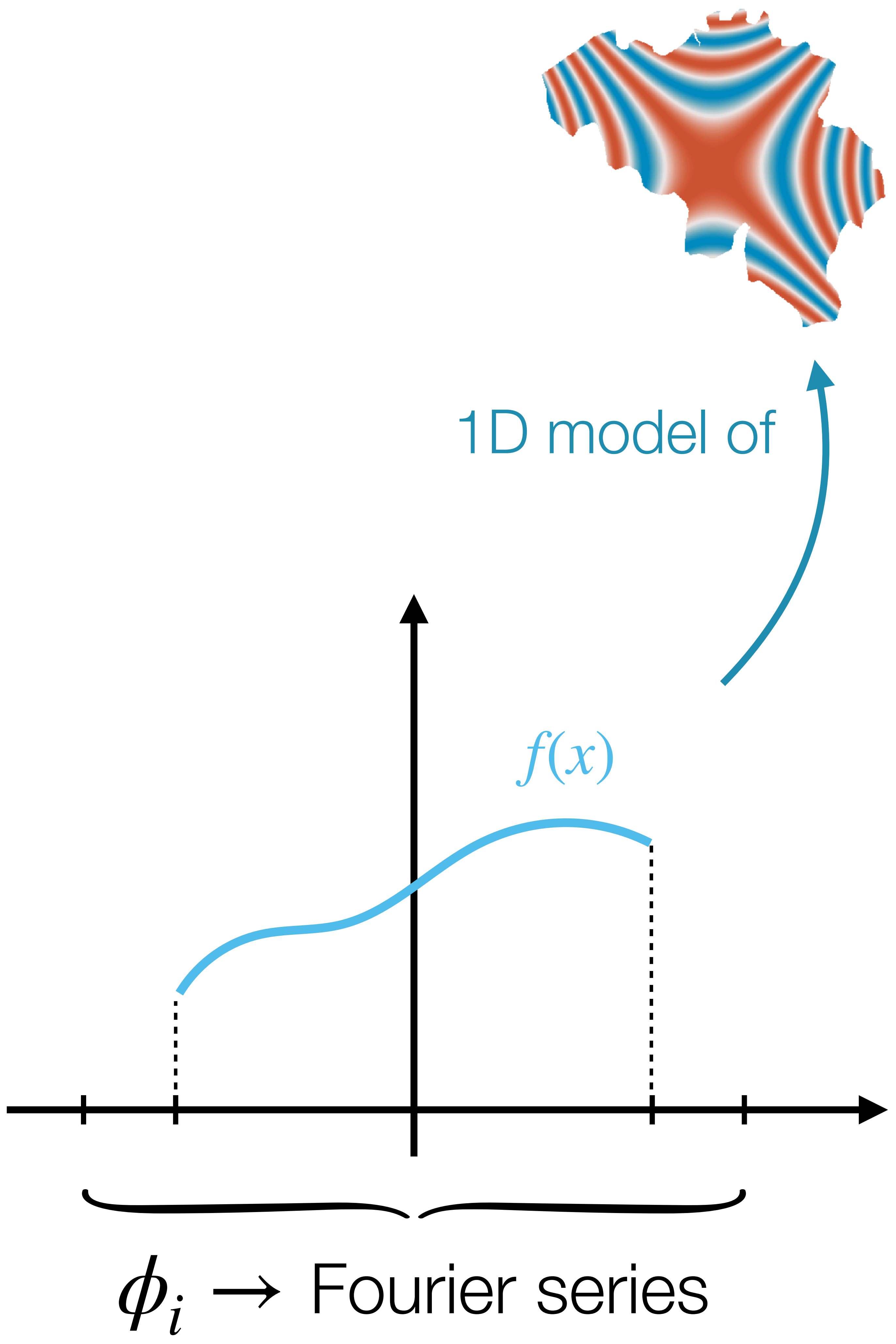

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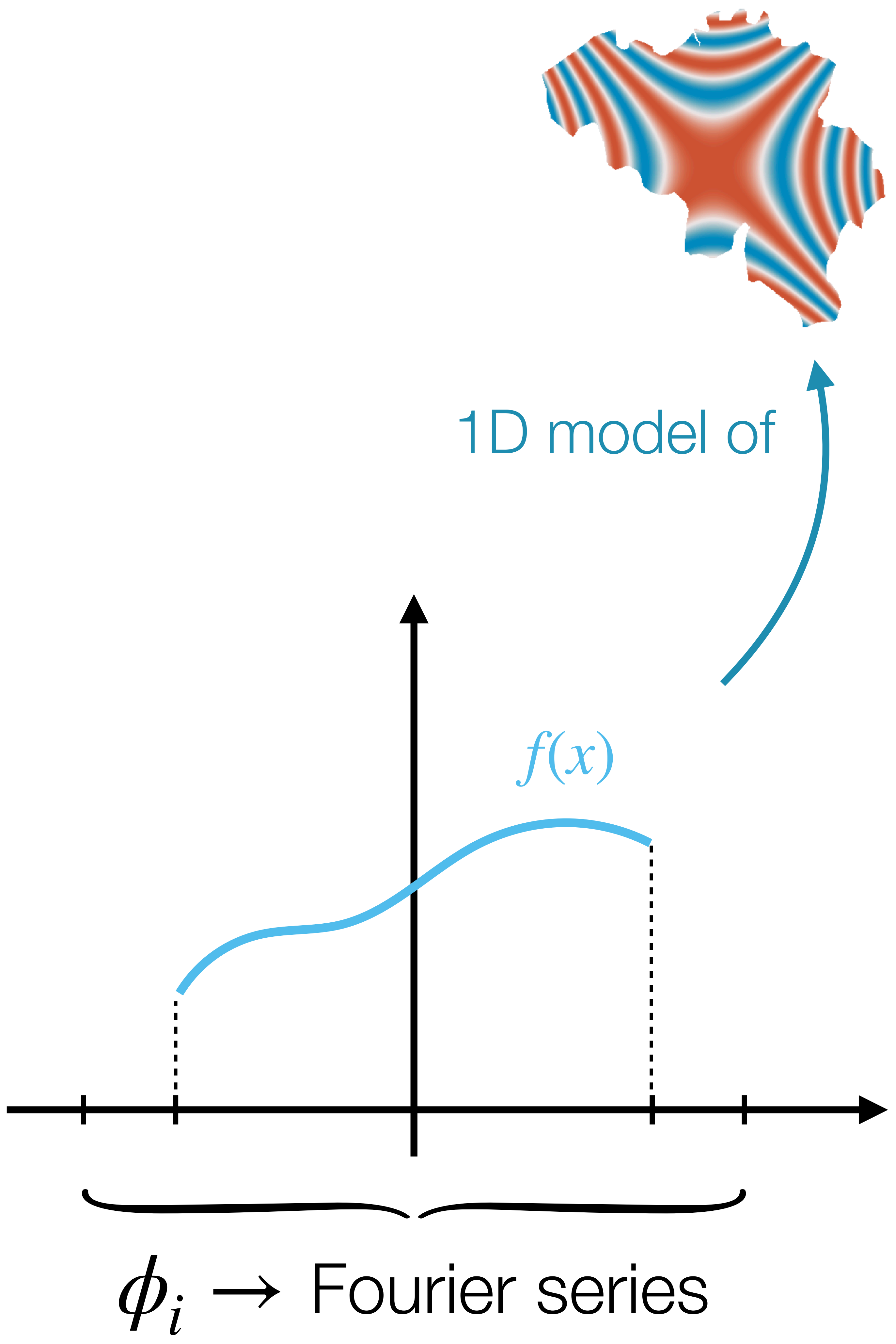

$k(x)$ grows much larger near the boundaries than the middle $\rightarrow$ we need to cluster points there

In reality, people compute stable least squares fits with a small number of uniformly random points

How is this possible? $\rightarrow k^{\epsilon}(x)$ is (approximately) uniform

(H. and Huybrechs, 2025)

- Approximation theory in finite precision
- An intuitive randomised sampling strategy
- **Efficient sampling for non-orthogonal bases**

For a given non-orthogonal basis $\{\phi_i\}_{i=1}^{n}\cdots$

We can construct an efficient sampler $\mathscr{M}$ using

$$k^{\epsilon}(x) = \Phi(x)^{*}\,(G + \epsilon^{2}I)^{-1}\,\Phi(x)$$

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We can approximate the Gram matrix using

$$(G)_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2} \approx \langle \mathscr{M}\phi_i, \mathscr{M}\phi_j \rangle_2$$

we don't know G…

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A “chicken or the egg” problem

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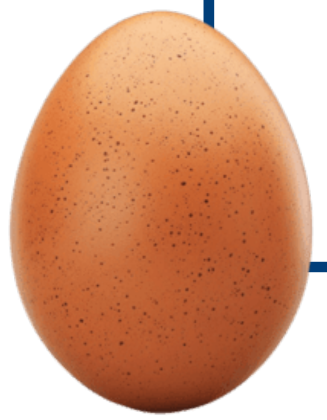

we don’t know an efficient $\mathscr{M}$…

Brute force approach

(Dolbeault and Cohen, 2022)

- Approximate G using a possibly huge number of uniformly random points
- Compute $m = \mathcal{O}(n \log(n))$ good samples for function approximation using Christoffel sampling

→ Good if the main cost lies in evaluating the functions to be approximated (i.e., approximating G is considered an “offline cost”)

Refinement-based Christoffel sampling

(H. and Adcock, 2025)

Consider $m = \mathcal{O}(n \log(n))$ samples drawn from

$d\mu = dx$ (uniform sampling)

Refinement-based Christoffel sampling

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Approximate the Gram matrix

$$(G)_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2} \approx \langle \mathcal{M}\phi_i, \mathcal{M}\phi_j \rangle_2 = (\widetilde{G}^{(1)})_{i,j}$$

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Consider $m = \mathcal{O}(n \log(n))$ samples drawn from

$d\mu = w\,dx$ where $w \propto \Phi(x)^*(\widetilde{G}^{(1)} + \epsilon^2 I)^{-1}\Phi(x)$

Approximate the Gram matrix

$$(G)_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2} \approx \langle \mathcal{M}\phi_i, \mathcal{M}\phi_j \rangle_2 = (\widetilde{G}^{(1)})_{i,j}$$

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approximation to k^ϵ

Consider $m = \mathcal{O}(n\log(n))$ samples drawn from

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iteration I:

Consider $m = \mathcal{O}(n\log(n))$ samples drawn from

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Approximate the Gram matrix

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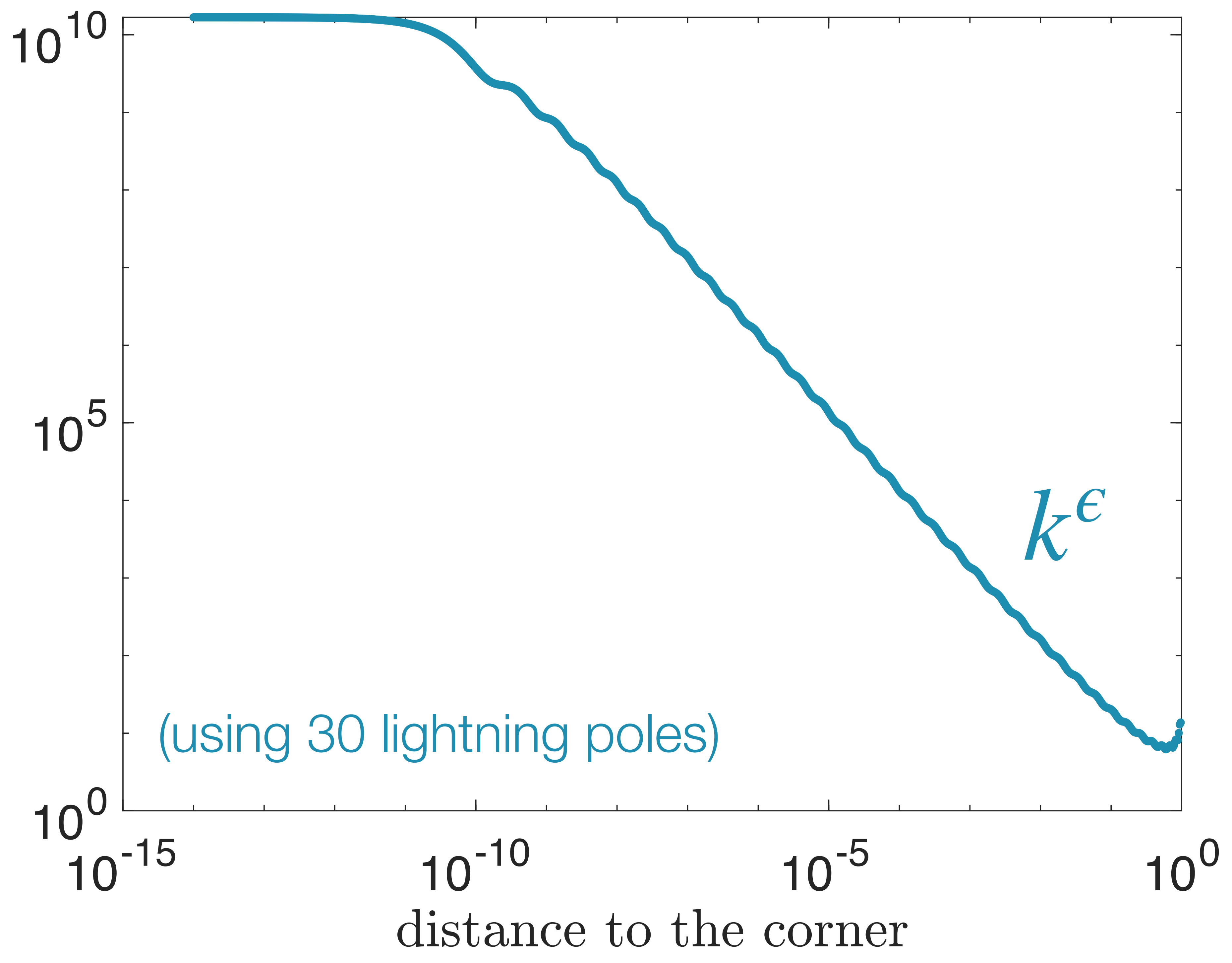
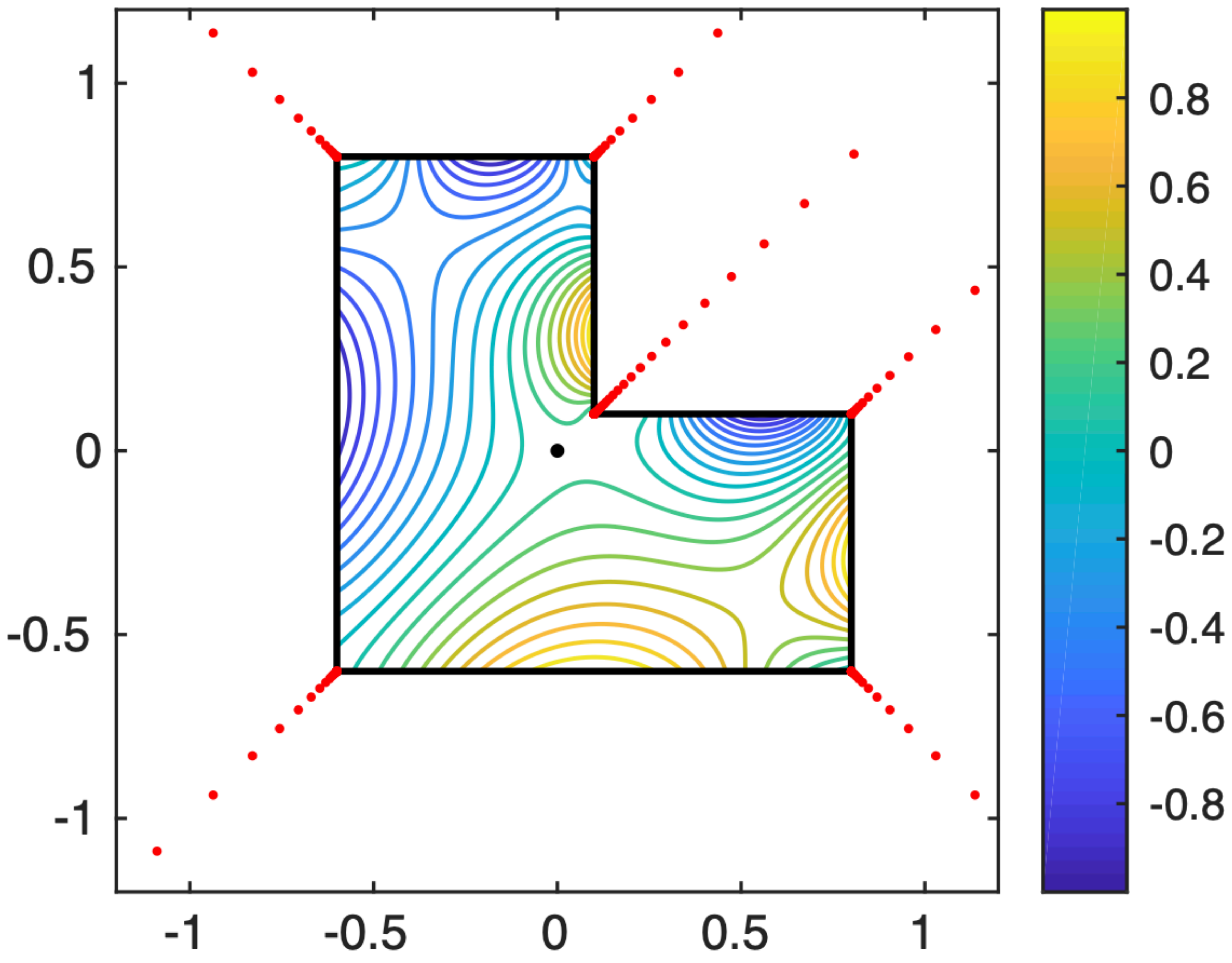
(disclaimer: this is a slight simplification)

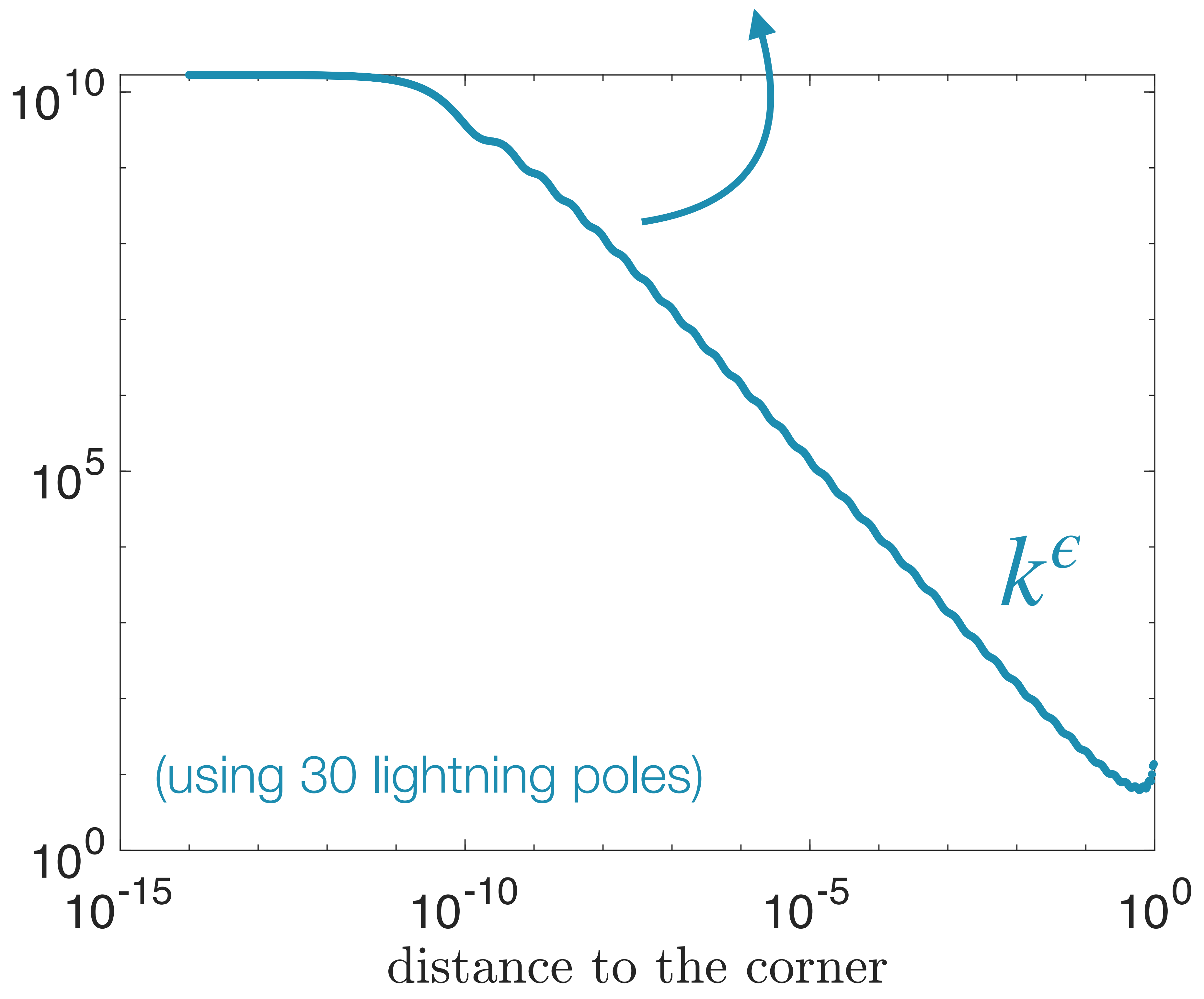


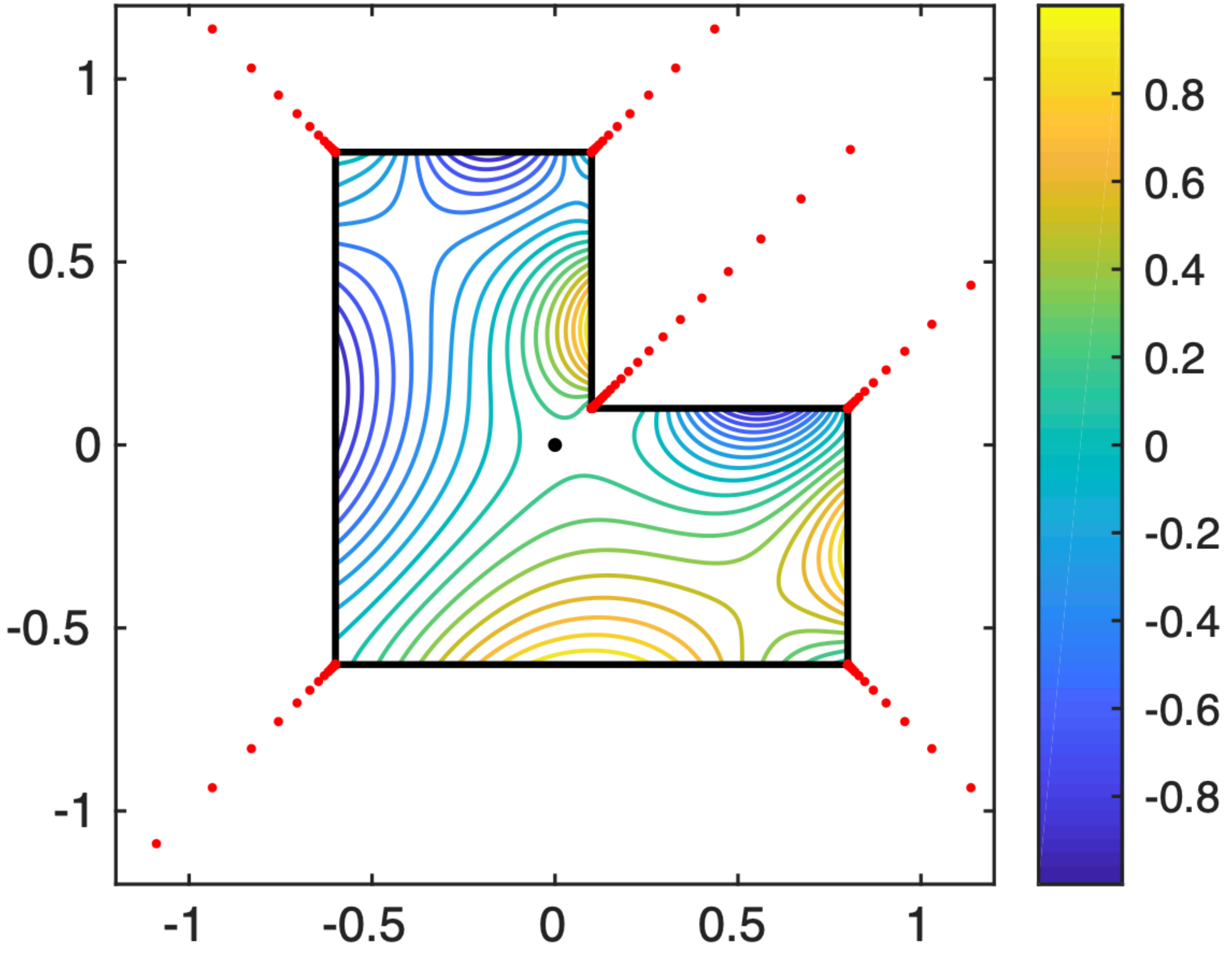
10^{10}
10^{5}
10^{0}
k^{ϵ}
(using 30 lightning poles)
10^{-15}
10^{-10}
10^{-5}
10^{0}
distance to the corner

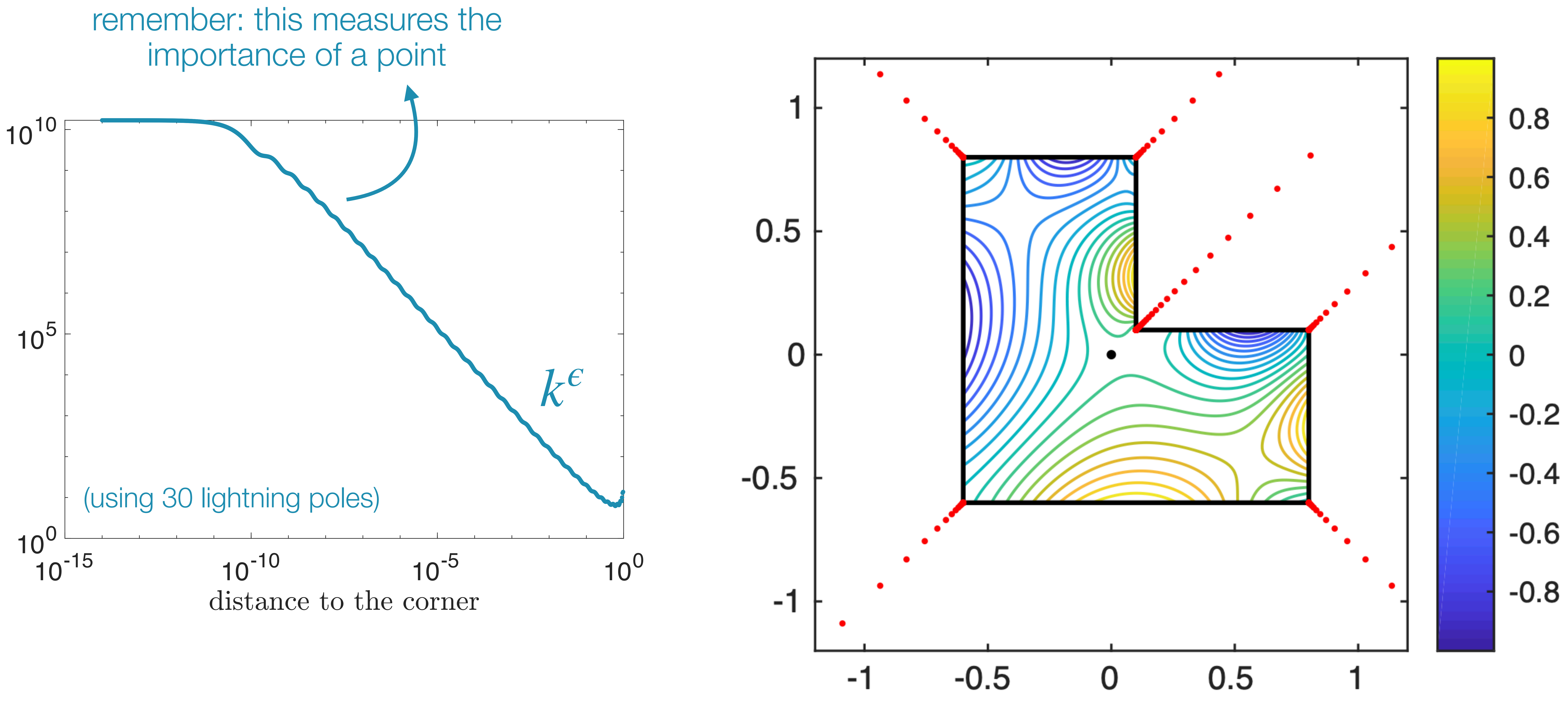


1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

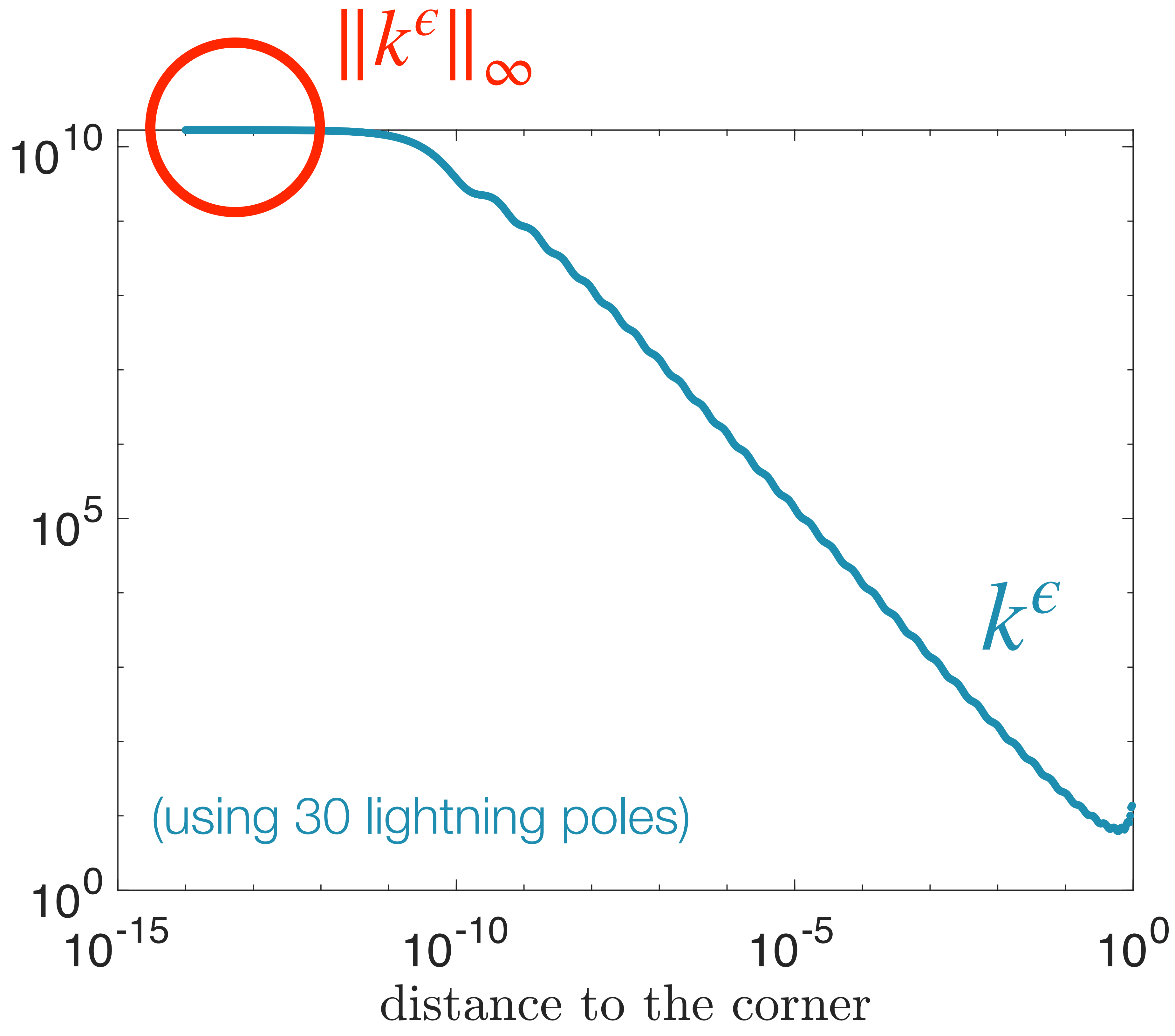
remember: this measures the importance of a point





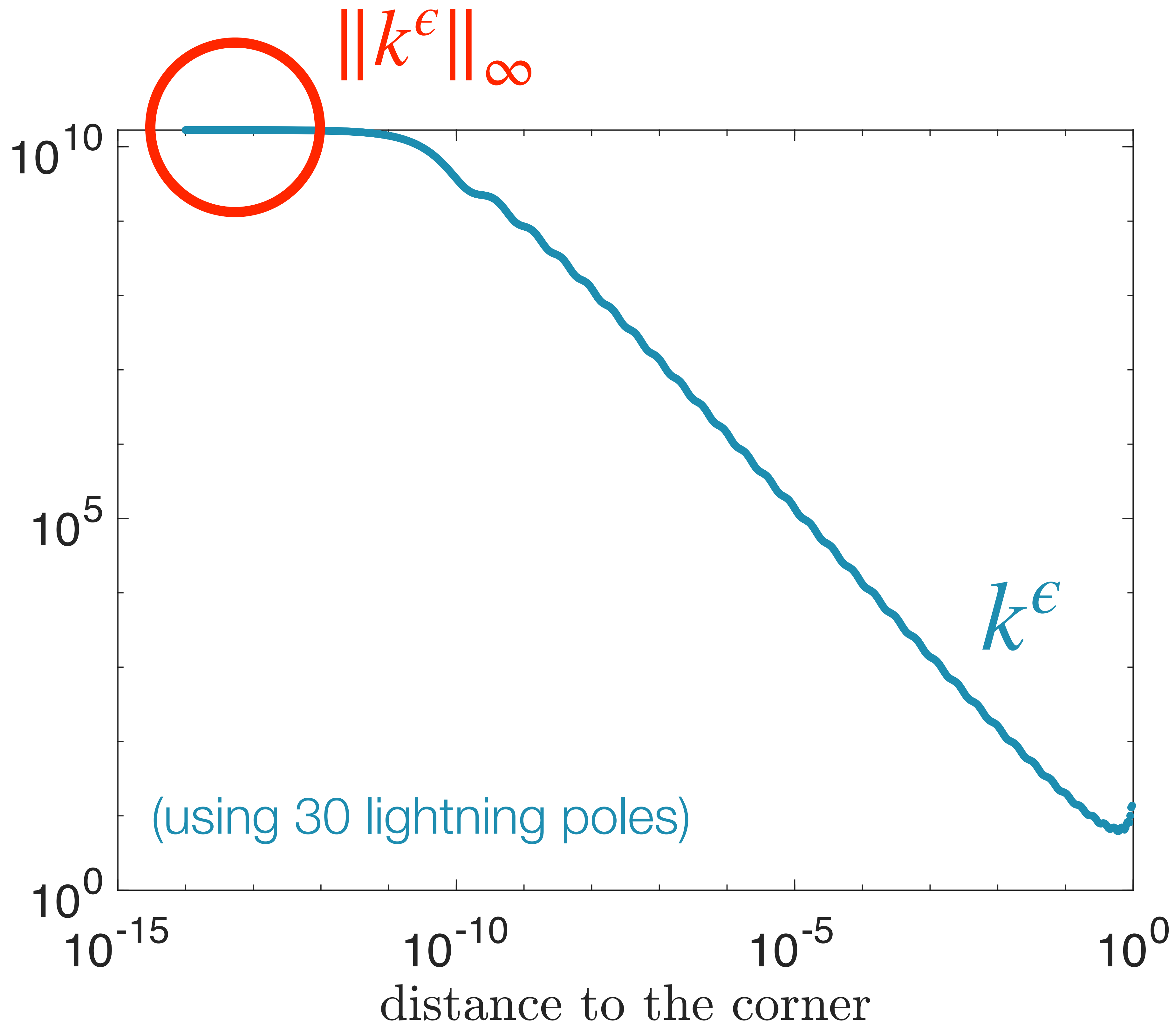


$\Rightarrow$ sample points should be exponentially clustered towards the corners



(Dolbeault and Cohen, 2022)

G should be computed using $\mathcal{O}(\|k^\epsilon\|_\infty \log(n))$ uniformly random sample points



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Converges in $\mathcal{O}(\log \|k^\epsilon\|_\infty)$ iterations and uses $\mathcal{O}(n \log(n))$ samples per iteration

Conclusions

- Non-orthogonal bases require approximation theory "in finite precision"

$$\left\| \mathcal{T}\tilde{c}_d - f \right\|_{L^2(X)} \lesssim \min_{c\in\mathbb{C}^n} \left\| \mathcal{T}c - f \right\|_{L^2(X)} + \epsilon\|c\|_2$$

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- The inverse Christoffel function quantifies the importance of each point for discrete approximation
- One can define a numerical Christoffel function that takes into account the effects of finite precision
- Refinement-based Christoffel sampling is an efficient algorithm for generating samples when using a non-orthogonal basis

References

Herremans, A. & Huybrechs, D. (2025). **Sampling theory for function approximation with numerical redundancy.** *preprint*, arXiv:2501.07470.

Herremans, A. & Adcock, B. (2025). **Refinement-based Christoffel sampling for least squares approximation in non-orthogonal bases.** *preprint*, arXiv:2510.08461.

Herremans, A., Huybrechs, D., & Trefethen, L. N. (2023). **Resolution of singularities by rational functions.** *SIAM J. Numer. Anal.*, 61(6), 2580-2600.

<u>More on the influence of finite precision</u>

Adcock, B. & Huybrechs, D. (2019). **Frames and numerical approximation.** *SIAM Rev.*, 61(2), 443-473.

Adcock, B. & Huybrechs, D. (2020). **Frames and numerical approximation II: generalized sampling**. *J. Fourier Anal. Appl.*, 26(6), 87.

<u>More on Christoffel sampling</u>

Cohen, A. & Migliorati, G. (2017). **Optimal weighted least-squares methods**. *SMAI J. Comput. Math.*, *3*, 181-203.

Dolbeault, M. & Cohen, A. (2022). **Optimal sampling and Christoffel functions on general domains**. *Constr. Approx.*, *56*(1), 121-163.

<u>More on the example</u>

Gopal, A. & Trefethen, L. N. (2019). **Solving Laplace problems with corner singularities via rational functions**. *SIAM J. Numer. Anal.*, 57(5), 2074-2094.